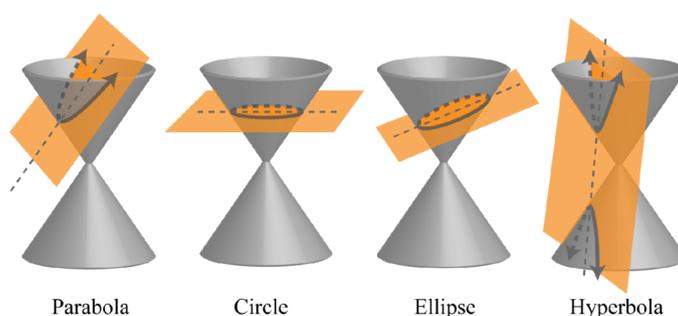


## 1 Introduction

In this project, we turn our attention to *conic sections*. [Apollonius of Perga](#) (c. 262 BC – c. 190 BC), a Greek geometer and astronomer, wrote eight books on *Conics*, influencing many later scholars. The theory of conics was taken up again by [Johannes Kepler](#) (27 Dec. 1571 – 15 Nov. 1630), a German mathematician, astronomer and astrologer, famous for his work in optics and [Kepler’s laws of planetary motion](#).

**Definition.** A *conic section* is a curve obtained by the intersection of the surface of a circular cone and a plane (not passing through the vertex, also called the *apex*, of the cone).



If the cone is cut by a plane that has the same slope as the cone (see picture above), then the intersection is a *parabola*; if the slope of the plane is less steep, then the intersection is an *ellipse* (and, as special case, if the plane is horizontal, a *circle*); and if the slope of the plane is steeper, then the intersection is a *hyperbola*.

Since the circular cone can be described as the set of points  $(x, y, z)$  in three-dimensional space that satisfy  $x^2 + y^2 = \alpha z^2$  for an appropriate  $\alpha > 0$  (note that  $\arctan(\alpha)$  is the slope of the cone), one can check that all these conic sections can be described as points  $(x, y)$  in the plane that satisfy some quadratic equation  $ax^2 + bxy + cy^2 + dx + ey + f = 0$  for some appropriate constants  $a, b, c, d, e, f$ ; in fact, by choosing the coordinate system appropriately (i.e., by rotating and translating the curves here appropriately, i.e., by looking at a congruent copy of the curve), we actually only<sup>1</sup> have to consider the three equations  $\alpha x^2 + \beta y^2 = 1$  which describes an ellipse,  $\alpha x^2 - \beta y^2 = 1$  which describes a hyperbola, and  $\alpha x^2 = y$  which describes a parabola, for some positive constants  $\alpha$  and

<sup>1</sup>We are discarding cases here where the quadratic equation reduces to equations of lines, or yields only a point or even the empty set (i.e., has no solution) or the whole plane.

$\beta$ . While this classification with respect to quadratic equations is useful, there other – more geometrical – ways to describe these curves.

Our goal is to explore these curves, the conic sections, using GeoGebra. We are interested in their properties, properties they share and that distinguish them from each other, as well as the properties that makes them interesting for applications in physics, astronomy and technical fields. You find the conic sections in the following menu in GeoGebra:



## 2 Parabola

In GeoGebra, we construct a parabola by choosing a point, called the *focus*, together with a line, the *directrix*. Among other things, we will discuss the significance of these two objects here, but first we construct a parabola: While you can choose any point and line (no matter of what slope), it might be useful to choose a point  $F = (0, y_F)$  on the  $y$ -axis as focus and the  $x$ -axis as directrix.

- We first explore how this construction and the quadratic equation  $y = \alpha x^2 + c$  with some parameters  $\alpha$  and  $c$  are related: Check that the vertex (i.e., the extremal point, thus the minimum or maximum) of the parabola is given by the midpoint between the origin  $(0, 0)$  and  $F$ . Thus, express the constant  $c$  in terms of  $y_F$ . Can you also express the parameter  $\alpha$  in terms of  $y_F$ ?
- We have just seen that the vertex of the parabola has the property that it has its distance to the focus  $F$  is the same as its distance to the directrix. We now check that this is true for any point on the parabola: Attach a point  $P$  to the parabola, and check that the distance  $FP$  and the distance from  $P$  to the directrix (your  $x$ -axis) is the same.  
(*Hint: There are several ways to do this; a nice geometrical way uses a circle!*)

Underlying this observation is the following geometrical definition of the parabola.

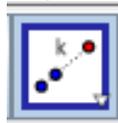
**Definition.** A *parabola* is the *equidistant set* (or *midset*) to a fixed point (the *focus*) and a line (the *directrix*) not containing this point; i.e., the parabola is the *locus*<sup>2</sup> of those points  $(x, y)$  in the plane that have the same distance from the focus and the directrix.

- Choose another point, say  $G$ , on the  $y$ -axis, and draw a second parabola with focus  $G$  and again with the  $x$ -axis as directrix. We now check that the two parabolas

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<sup>2</sup>A *locus* is a set of points whose location is determined by one or more specified conditions. For the parabola, this condition is the equal distance to the focus and the directrix.

have exactly the same shape (i.e., they are *similar*): Using “Dilate from Point”, show that any two parabolas are only scaled copies of each other.



- We now consider the original parabola with focus  $F$  and point  $P$  on it again. Find the tangent  $t$  to the parabola at  $P$ , the line  $p$  that is perpendicular to the directrix and passes through  $P$ , and the line  $f$  joining  $F$  and  $P$ . Measure the angle at  $P$  between  $p$  and the tangent  $t$  and the angle between  $f$  and the tangent  $t$ , and make note of your observation.

With the observation you just made, you should be able to find an explanation for the following application of parabolas: To observe distant stars, astronomers use mirrors shaped like (circular) *paraboloids*, which are parabolas rotated around their axes. What is the reason they use a paraboloid mirror? Flashlights (or the lights in a car) also have a mirror shaped like (circular) paraboloids, with the lightbulb located at the focus. What is the reason here?

- We can use a slider to move the point  $P$  along the parabola. Recalling the command **Curvature**, construct the *osculating circle* on the parabola at  $P$ . Thus, as you move the slider the point  $P$  together with the osculating circle should change.

**Definition.** The locus of all the centers of the osculating circles, is called the *evolute* of the given curve.

We now let GeoGebra plot the evolute of our parabola: Using “Locus”, select the center of the osculating circle and your slider to obtain this evolute.



- As a last point, we look at 3-dimensional generalizations of the parabola, the *paraboloids*. They actually come in two shapes: *elliptic paraboloids* (with the special case of *circular paraboloids*) and *hyperbolic paraboloids*. An example for the former is the surface with equation  $x^2 + y^2 = z$ , while an example for the latter is  $x^2 - y^2 = z$ . Plot these *quadric surfaces* in GeoGebra using the 3D grapher (make sure that you have GeoGebra 5). By looking at their shape, can you explain their names?

### 3 Ellipse

In GeoGebra, we construct an ellipse by choosing two points, called the *foci*, together with a point on the ellipse. Among other things, we will discuss the significance of the two *foci* here, but first we construct an ellipse. It might be useful to choose the two foci  $F_1$  and  $F_2$  to lie on a horizontal line, e.g., the  $x$ -axis (symmetrical around the origin). Then choose a third point  $A$  through which the ellipse should pass (you can immediately hide this point  $A$  again – you only need it when you want to change the ellipse itself).

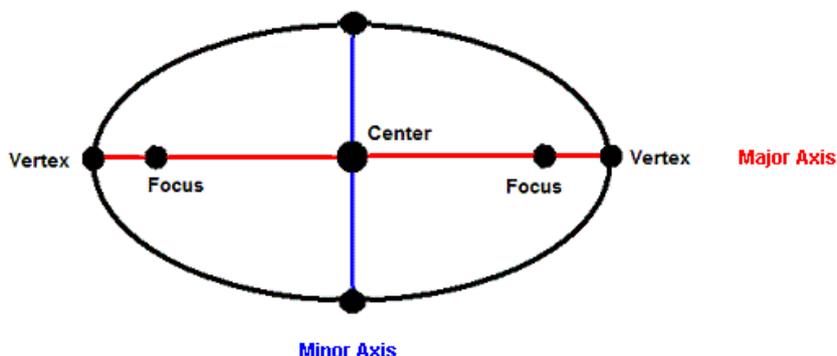
- Attach a point  $P$  to the ellipse, and measure the distances  $F_1P$  and  $F_2P$ . Observe that as you move  $P$  around the ellipse, the sum of the two distances is constant.

We have observed the following geometrical definition of an ellipse.

**Definition.** The ellipse is the locus of those points of the plane whose sum of distances to two fixed points  $F_1$  and  $F_2$  (the *foci*) is constant.

Note that an ellipse can be constructed using two pins and a string (the so-called “gardener’s method”). Explain!

- We need some more vocabulary: The *center* is the midpoint between the two foci. Ellipses have two axes of symmetry; the larger one, called the *major axis* of an ellipse, is the segment through the two foci connecting two antipodal points (the *vertices*) of the ellipse, the smaller one, called the *minor axis* of an ellipse, is the segment perpendicular to the major axis through the center connecting antipodal points on the ellipse. Half the respective segment from the center to the point on the ellipse are called the *semi-major axis* and *semi-minor axis*, respectively. The length of the semi-major axis is usually denoted  $a$  (so the major axis has length  $2 \cdot a$ ), the length of the semi-minor axis is usually denoted  $b$ . Previously, we looked at the sum of the distances  $F_1P$  and  $F_2P$ : Can you express this sum in terms of  $a$  and/or  $b$ ? Also, measure the distance from the center to one of the foci  $F_1$  or  $F_2$ , this length is called *linear eccentricity*  $c$ . Can you explain why  $a^2 = b^2 + c^2$ ?



- The<sup>3</sup> *eccentricity*  $e$  of the ellipse is given by  $\frac{c}{a}$ , i.e., the ratio of the distance from the center to a focus to the length of the semi-major axis. Measure the distance from your focus  $F_1$  to its nearest vertex (see picture above). We call that distance  $p$ . Now, construct a circle of radius  $\frac{p}{e}$  around this vertex, and denote the point of intersection with the line through the two foci (i.e., along the major axis) on the outside of the ellipse by  $D$ . Draw the line perpendicular to the major axis through this point  $D$ . This perpendicular line is called a *directrix* of the ellipse. We now measure the distance  $F_1P$  from the focus  $F_1$  to the point  $P$  on the ellipse, and compare it with the distance of  $P$  from the directrix (*Hint: Construct a perpendicular line to the directrix through  $P$* ) by looking at the ratio of these two distances. Can you give a second geometrical definition of an ellipse? (*Hint: “The locus of all points for which the ratio of ...”*)
- We now only consider the ellipse with the two foci  $F_1$  and  $F_2$  together with a point  $P$  on the ellipse. Construct the tangent  $t$  to the ellipse at  $P$ . Compare the angles between  $t$  and the line through  $P$  and  $F_1$ , and between  $t$  and the line through  $P$  and  $F_2$ .

What does this last observation mean in terms of rays of light emitted from a lightbulb located at one of the foci? For another application of ellipses, read [Kepler’s laws of planetary motion](#).

Finally, do one of the following two constructions:

- Find the *evolute* of your ellipse, i.e., use an (angular) slider to move the point  $P$  around the ellipse, and then construct the *osculating circle* to the ellipse at  $P$  (which should change as you move the point  $P$  around). Then use GeoGebra to plot the evolute.
- The ellipse can also be obtained as *equidistant set* (or *midset*) to one of the two foci and a circle centered at the other focus that contains the first focus in its inside. Find the radius of this circle (and demonstrate, using GeoGebra, this property).

We note the ellipse you constructed here is congruent to the ellipse that satisfies the equation  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ . In three dimensions, this generalizes to the *ellipsoid* with equation  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$  that has *semi-principal axes* of length<sup>4</sup>  $a, b, c$ . Ellipsoids occur frequently in physics as well as statistics, so draw a few of them in GeoGebra using the 3D grapher (make sure that you have GeoGebra 5). What do you get if  $a = b = c$ ? What if  $a = b \neq c$ ?

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<sup>3</sup>Don’t use the letter “ $e$ ” in GeoGebra though, since that is reserved for Euler’s constant.

<sup>4</sup> These lengths  $a, b, c$  are not the same as the length of the semi-major axis, the distance between the center and a focus, and the length of the semi-minor axis we found above – they are new variables here

## 4 Hyperbola

In GeoGebra, we construct a hyperbola similar to the ellipse, i.e., we first choose two points, called the *foci*  $F_1$  and  $F_2$ , together with a point on the hyperbola. We will thus proceed similar to the last section. In particular, it might be useful to choose the two foci  $F_1$  and  $F_2$  to lie on a horizontal line, e.g., the  $x$ -axis. Again, the third point  $A$  through which the hyperbola passes is only needed if we want to change the hyperbola. We note that the hyperbola actually consists of two curves, called the two *branches* of the hyperbola.

- Attach a point  $P$  to (one of the branches of) the hyperbola, and measure the distances  $F_1P$  and  $F_2P$ . Observe that as you move  $P$  around the hyperbola, the difference of the two distances is constant.

We have observed the following geometrical definition of a hyperbola.

**Definition.** The hyperbola is the locus of those points of the plane whose difference of distances to two fixed points  $F_1$  and  $F_2$  (the *foci*) is constant.

- We use the same vocabulary as for ellipses: The midpoint between the two foci is called the *center* of the hyperbola. The intersection of the line through the foci and the hyperbola yield the two vertices of the hyperbola. We denote the distance from the center to a focus by  $c$ , and the distance from the center to a vertex by  $a$  (note that  $c > a$  while for the ellipse we had  $c < a$ ). If you constructed your hyperbola symmetrically around the origin, your foci should be at  $(-c, 0)$  and  $(c, 0)$ , the center at the origin  $(0, 0)$  and the vertices at  $(-a, 0)$  and  $(a, 0)$ . We calculate a number  $b$  by  $c^2 = a^2 + b^2$ , i.e.,  $b = \pm\sqrt{c^2 - a^2}$ . Now draw the two lines  $y = \pm\frac{b}{a}x$ . Can you figure out what their significance is?
- The<sup>5</sup> *eccentricity*  $e$  of the hyperbola is given by  $\frac{c}{a}$ , i.e., the ratio of the distance from the center to a focus to the distance from the center to a vertex. We now construct one directrix, say the one belonging to the branch on the left with focus  $F_1 = (c, 0)$  with  $c > 0$ : In this case, the directrix is the vertical line (parallel to the  $y$ -axis, perpendicular to the  $x$ -axis) through the point  $(\frac{a}{e}, 0)$ . To confirm this, measure the distance  $F_1P$  from the focus  $F_1$  (on the left) to the point  $P$  on the (the left branch of the) hyperbola, and compare it with the distance of  $P$  from the directrix (*Hint: Construct a perpendicular line to the directrix through  $P$* ) by looking at the ratio of these two distances. Can you give a second geometrical definition of an hyperbola?  
(*Hint: "The locus of all points for which the ratio of ..."*)
- We now only consider the hyperbola with the two foci  $F_1$  and  $F_2$  together with a point  $P$  on one of the branches of the hyperbola. Construct the tangent  $t$  to the

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<sup>5</sup>Don't use the letter " $e$ " in GeoGebra though, since that is reserved for Euler's constant.

hyperbola at  $P$ . Compare the angles between  $t$  and the line through  $P$  and  $F_1$ , and between  $t$  and the line through  $P$  and  $F_2$ .

What does this last observation mean in terms of rays of light emitted from a lightbulb located at one of the foci? We also mention that interplanetary satellites like **New Horizon** (remember last year's flyby of Pluto?) approximately follow a (branch of a) hyperbola with the planet at the focus.

Then, do one of the following two constructions:

- Find the *evolute* of your hyperbola, i.e., use a slider to move the point  $P$  along the hyperbola, and then construct the *osculating circle* to the hyperbola at  $P$  (which should change as you move the point  $P$  around). Then use GeoGebra to plot the evolute.
- The hyperbola can also be obtained as *equidistant set* (or *midset*) to one of the two foci and a circle centered at the other focus with the first focus outside this circle. Find the radius of this circle (and demonstrate, using GeoGebra, this property).

Finally, we look at 3-dimensional generalizations.

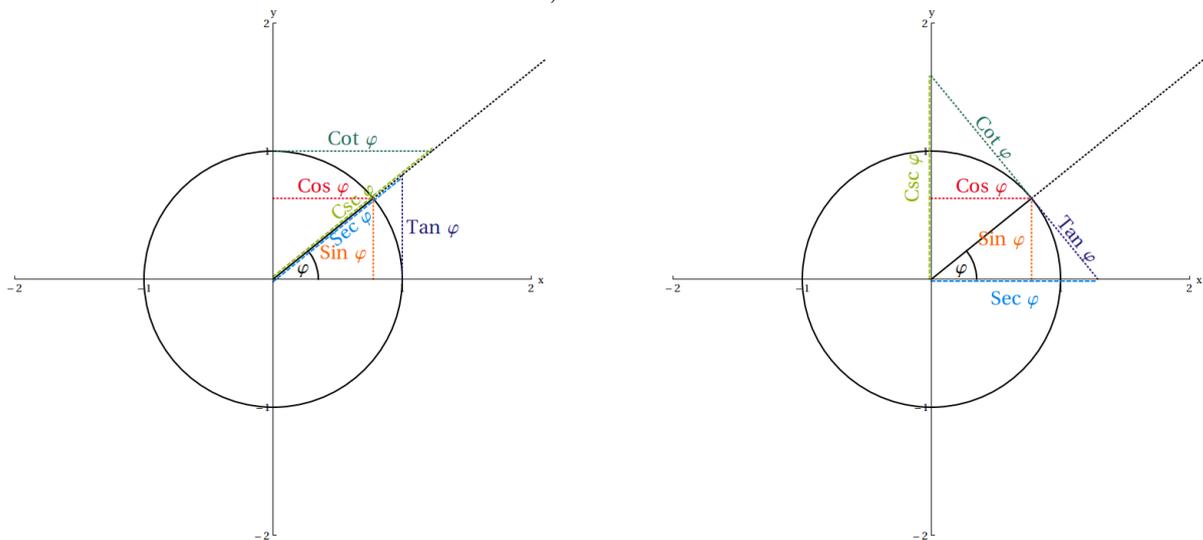
- As in the case of the parabola, there are two shapes of *hyperboloids*: the (*elliptic*) *hyperboloid of one sheet* (with the special case of *circular hyperboloid of one sheet*) and the (*elliptic*) *hyperboloid of two sheets*. An example for the former is the surface with equation  $x^2 + y^2 - z^2 = 1$ , while an example for the latter is  $x^2 + y^2 - z^2 = -1$ . Plot these *quadric surfaces* in GeoGebra using the 3D grapher (make sure that you have GeoGebra 5). By looking at their shape, can you explain their names? One-sheeted hyperboloids are also popular in architecture (e.g., cooling towers) because that are also a *doubly ruled surface*. Explain why this property is useful in constructing these hyperboloids.

## 5 Extra: Trig Functions and Hyperbolic Trig Functions

You might be familiar with the trigonometric functions sine, cosine, tangent, cotangent, secant and cosecant and how to define them geometrically using the unit circle  $x^2 + y^2 = 1$ . Let us repeat this here (also see [here](#)):

- Let  $A$  be a point on the unit circle,  $O = (0, 0)$  the origin and  $P = (1, 0)$ . With such a point  $A$  on the unit circle, we associate the angle  $\varphi = \angle POA$  in counter-clockwise direction (thus  $0^\circ \leq \varphi < 360^\circ$ ). However, instead of measuring this angle in degrees, we use radian which on the unit circle is simply the arc length of the arc between  $P$  and  $A$ . In these units, we have  $0 \leq \varphi < 2\pi$ .

- For simplicity, we will just consider angles up to the right angle here, i.e.,  $0 \leq \varphi \leq \frac{\pi}{2}$ . So construct a unit circle, attach a point  $A$  to it, measure the corresponding angle  $\varphi$  (turn that measurement into radians). Alternatively, you could also define an angular slider and control the point  $A$  on the unit circle by that slider.
- Check that the coordinates of  $A$  are given by  $(\cos \varphi, \sin \varphi)$ .
- Draw the ray  $\overrightarrow{OA}$  from the origin  $O$  through  $A$ . We now give a geometric interpretation of tangent and cotangent: Draw the parallel line to the  $y$ -axis through the point  $P = (1, 0)$  (note that this is a vertical tangent to the unit circle), and find the intersection point of this line and the ray  $\overrightarrow{OA}$ . Check that the coordinates of this intersection point are  $(1, \tan \varphi)$ . Similarly, draw the horizontal line parallel to the  $x$ -axis through the point  $(0, 1)$ . Again, find the intersection point of this line and the ray  $\overrightarrow{OA}$ , and check that its coordinates are  $(\cot \varphi, 1)$ .
- Finally, let us find secant and cosecant: Draw the tangent  $t$  to the unit circle at  $A$ . Find the intersection point of this tangent  $t$  and the  $x$ -axis and check that its coordinates are  $(\sec \varphi, 0)$ . Similarly, find the intersection point of this tangent  $t$  and the  $y$ -axis and check that its coordinates are  $(0, \csc \varphi)$ .
- The following two pictures depict where we can find trigonometric functions on the unit circle (we indicated a few more lengths that correspond to trigonometric functions we haven't checked here):



- Using the `Integral` respectively `IntegralBetween` command in GeoGebra, we check the following: The area of the region bounded by the  $x$ -axis, the unit circle and the ray  $\overrightarrow{OA}$  (i.e., the region of the circular sector with central angle  $\varphi$ ) equals  $\frac{1}{2}\varphi$ .

We now turn our attention to the **hyperbolic (trigonometric) functions** which appear frequently in physics and engineering.

- Draw the unit hyperbola  $x^2 - y^2 = 1$  in GeoGebra. Attach a point  $A$  to the right branch of this hyperbola, and draw the ray  $\overrightarrow{OA}$  from the origin  $O$  through  $A$ . We will only consider the case where  $A$  is in the first quadrant, i.e., both its coordinates are nonnegative.
- We first need something that is equivalent to the angle  $\varphi$  in the above trigonometric function case: Using the `Integral` respectively `IntegralBetween` command in GeoGebra, we find the area of the region bounded by the  $x$ -axis, the unit hyperbola and the ray  $\overrightarrow{OA}$ . We define the *hyperbolic angle*  $\theta$  as **twice** this area.
- Now we essentially proceed as before:
  - Check that the coordinates of the point  $A$  is given by  $(\cosh \theta, \sinh \theta)$ .
  - Construct the tangent to the hyperbola at  $(1, 0)$  (again, this is a vertical line, parallel to the  $y$ -axis), and its intersection point with the ray  $\overrightarrow{OA}$ . Check that the coordinates of this intersection point is  $(1, \tanh \theta)$ .
  - Construct horizontal line through  $(0, 1)$  (parallel to the  $x$ -axis), and its intersection point with the ray  $\overrightarrow{OA}$ . Check that the coordinates of this intersection point is  $(\coth \theta, 1)$ .
  - Finally find the tangent  $t$  on the hyperbola at  $A$ . Check that the intersection point of  $t$  and the  $x$ -axis is  $(\operatorname{sech} \theta, 0)$ , while the intersection point of  $t$  and the  $y$ -axis is  $(0, -\operatorname{csch} \theta)$ .

Can you explain why the inverse functions of the trigonometric functions are denoted  $\arcsin(x)$ ,  $\arccos(x)$ , etc., while the inverse hyperbolic functions are denoted  $\operatorname{arsinh}(x)$ ,  $\operatorname{arcosh}(x)$ , etc. (read: “area hyperbolic sine of  $x$ ”, “area hyperbolic cosine of  $x$ ”, etc.)?