Pisot substitutions (and adelic spaces)

(my view from the Bielefelder Alm)

Bernd Sing
Fakultät für Mathematik
Universität Bielefeld
Germany
sing@math.uni-bielefeld.de

Adeles

- Completions of an algebraic number field K: Local fields K_ν ($\nu \in J$).

- Places J: Archimedean absolute value $\Rightarrow$ infinite place (finitely many)
  Non-Archimedean absolute value $\Rightarrow$ finite place

- E.g.: $\mathbb{Q}$ $\Rightarrow$ $\mathbb{R}$ and $\mathbb{Q}_p$ ($p \in \mathbb{P}$)
  $\mathbb{Q}(\lambda)$, $d = [\mathbb{Q}(\lambda) : \mathbb{Q}]$ $\Rightarrow$ r times $\mathbb{R}$, s times $\mathbb{C}$ ($r + 2s = d$)
  and $\mathbb{Q}_\mathfrak{p}$ ($\mathfrak{p} \in \mathbb{P}_{\mathbb{Q}(\lambda)}$).

- Adele ring $\mathbb{A}_K$ of an algebraic number field K is the restricted direct product of the local fields of K, i.e.,

$$\mathbb{A}_K = \{(x_\nu)_{\nu \in J} \mid x_\nu \in K_\nu \text{ with } x_\nu \in \mathfrak{o}_\nu \text{ for all but finitely many } \nu\}$$

  (where $\mathfrak{o}_\nu$ denotes the ring of ν-adic integers, e.g., $\mathbb{Z}_p$).

Principal Lattice Transformations on $\mathbb{A}_K^d$

- Denote by $\mathbb{A}_K^d$ the space of d-dimensional adeles of K, i.e.,

  $\mathbb{A}_K^d = \{(x_\nu)_{\nu\in J} | x_\nu \in (K_\nu)^d$ with $x_\nu \in (\mathfrak{o}_\nu)^d$ for all but finitely many $\nu\} = (\mathbb{A}_K)^d$.

- Linear transformations on $\mathbb{A}_K^d$:
  $\boldsymbol{M} = (\boldsymbol{M}_\nu)_{\nu\in J}$, where $\boldsymbol{M}_\nu$ is a $d \times d$-matrix over K_ν with all coefficients in $\mathfrak{o}_\nu$ for all but finitely many $\nu \in J$.

- Lattice transformations on $\mathbb{A}_K^d$:
  $\boldsymbol{M} = (\boldsymbol{M}_\nu)_{\nu\in J}$, where $\det \boldsymbol{M}_\nu \neq 0$ and $|\det \boldsymbol{M}_\nu|_\nu = 1$ for all but finitely many $\nu \in J$.
  Matrix norm: $\|\boldsymbol{M}\| = \prod_{\nu\in J} |\det \boldsymbol{M}_\nu|_\nu$.

- Principal lattice transformations on $\mathbb{A}_K^d$:
  Let $\boldsymbol{M}$ be a non-singular $d \times d$-matrix over $K \quad \Rightarrow \quad (\boldsymbol{M})_{\nu\in J}$
  Then: $\|\boldsymbol{M}\| = 1$.

Substitution Matrix

- Can regard substitution matrix as principal lattice transformation on $\mathbb{A}_{\mathbb{Q}}^d$.
- Let $p(x) = \det(x \cdot \boldsymbol{E} - \boldsymbol{M})$ be the characteristic polynomial. Then, $p(x)$ has d distinct eigenvalues $\lambda_1^{(\nu)}, \ldots, \lambda_d^{(\nu)}$ in the algebraic closure K_ν^a for each $\nu \in J$.
- For each $\nu \in J$, take classes of algebraic conjugate roots. If such a class has $\tilde{d}$ members, then the associated minimal polynomial has degree $\tilde{d}$ and $K_\nu(\lambda)$ is a $\tilde{d}$-dimensional K_ν-vector space for each such member.
- If there are k such classes, take representatives $\lambda_i^{(\nu)}$ $(1 \le i \le k)$. Then the action of $\boldsymbol{M}$ on K_ν^d can be identified with the action of

$$T_\nu : \prod_{i=1}^{k} K_\nu(\lambda_i^{(\nu)}) \to \prod_{i=1}^{k} K_\nu(\lambda_i^{(\nu)}),\ T_\nu(x_1, \ldots, x_k) = (\lambda_1^{(\nu)} x_1, \ldots, \lambda_k^{(\nu)} x_k),$$

where $\prod_{i=1}^{k} \|\lambda_i^{(\nu)}\|_{K_\nu(\lambda_i^{(\nu)})} = |\det \boldsymbol{M}|_\nu$.

Substitution Matrix and Multiplication by λ

- Principal lattice transformation $(\boldsymbol{M})_{\nu\in J}$ on $\mathbb{A}_{\mathbb{Q}}^{d}$ can be identified with the multiplication map $T(x) = (\lambda \cdot x_{\nu'})_{\nu'\in J'}$ on $\mathbb{A}_{\mathbb{Q}(\lambda)}$, where $\lambda \mapsto (\lambda, \lambda, \ldots)$ is the diagonal embedding of (a) root of $p(x)$ into $\mathbb{A}_{\mathbb{Q}(\lambda)}$.

- Pisot-property: $\lambda = \lambda_1$ is expansion on $\mathbb{R}$, but its Galois conjugates act as contraction on $\mathbb{R}$ ($r-1$ times) respectively $\mathbb{C}$ (s times).

- On finite places: Can only be contracting on finitely many local fields, namely on extension(s) of K_ν if $|\det \boldsymbol{M}|_\nu < 1$.

$\rightsquigarrow$ Markov-partition: Expanding and contracting "eigenspace of $\boldsymbol{M}$".

$\rightsquigarrow$ From model set theory: Product of all contracting spaces is "internal space" H, a.k.a. the space for the "geometric realization" (where the "window", respectively the "Rauzy fractal" lives).

Substitution

$\frac{1}{\lambda}$ is expansion (contraction) iff λ is contraction (expansion) on a given space.

On $\mathbb{R}$, we can define a self-replicating tiling by intervals of lengths ℓ_i ($1 \leq i \leq d$) by

$$\mathbb{R} = \bigcup_{i=1}^{d} [0\,, \ell_i] + \Lambda_i.$$

Here, $(\ell_1, \ldots, \ell_d) \in (\mathbb{Q}(\lambda_1))^d$ is an eigenvector of $\boldsymbol{M}$ to λ_1 and Λ_i is given by

$$\Lambda_i = \bigcup_{j=1}^{d} \lambda_1\, \Lambda_j + A_{ij}.$$

The set A_{ij} ($\operatorname{card} A_{ij} = M_{ij}$) is determined by the substitution. More precisely, A_{ji} can be obtained from ($i \mapsto j_1 \ldots j_k$)

$$[0\,, \ell_i] \xrightarrow{\lambda_1} [0\,, \lambda_1 \cdot \ell_i] = 0 + [0\,, \ell_{j_1}] \cup \ell_{j_1} + [0\,, \ell_{j_2}] \cup \ldots \cup (\lambda_1 \cdot \ell_i - \ell_{j_k}) + [0\,, \ell_{j_k}].$$

Observe: $[0\,, \ell_i] = \bigcup_{j=1}^{d} \frac{1}{\lambda_1}\, [0\,, \ell_j] + \frac{1}{\lambda_1}\, A_{ji}$.

Dual Tilings

On $\mathbb{R}$: $$\varLambda_i = \bigcup_{j=1}^{d} \lambda_1 \varLambda_j + A_{ij}$$

$$[0\,,\ell_i] = \bigcup_{j=1}^{d} \tfrac{1}{\lambda_1}\,[0\,,\ell_j] + \tfrac{1}{\lambda_1}\,A_{ji}$$

On H: $$\varOmega_i = \bigcup_{j=1}^{d} \lambda^\star\, \varOmega_j + A_{ij}^\star$$

$$\varGamma_i = \bigcup_{j=1}^{d} \left(\tfrac{1}{\lambda}\right)^\star\, \varGamma_j + \left(\tfrac{1}{\lambda}\right)^\star\, A_{ji}^\star$$

- IFS $\rightsquigarrow$ unique non-empty compact solution
- Expansions $\rightsquigarrow$ Problem: How to choose start values properly!?!

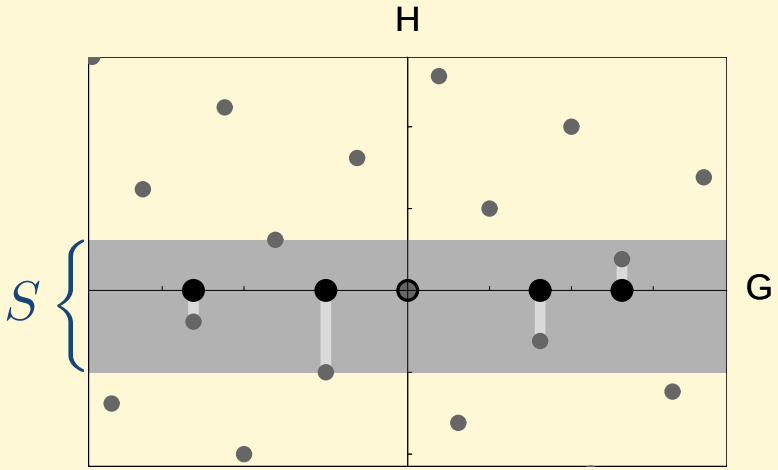
In fact: We know which solutions we want to ensure pure pointedness, namely (in terms of model sets) $\varLambda_i = \Lambda(\overset{\ominus}{\varOmega}_i)$ respectively $\varGamma_i = \Lambda([0\,,\ell_i[)$.

Question: Are (the self-replicating coverings with the same a.e. constant covering degree) $\bigcup_i \varLambda_i + [0\,,\ell_i]$ and $\bigcup_i \varOmega_i + \varGamma_i$ tilings?

Model Sets

Cut and project scheme:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \text{ dense} \\ L & \overset{\text{bijective}}{\longleftrightarrow} & \tilde{L} & \longrightarrow & L^\star \end{array}$$



Model set: $\varLambda = \Lambda(S) = \{x \in G \mid \pi_2(x) \in S\}$

Here: $G = \mathbb{R}$, $H = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{Q}_{\mathfrak{p}_1} \times \cdots \times \mathbb{Q}_{\mathfrak{p}_k}$, $L = \langle \ell_1, \ldots \ell_d \rangle_{\mathbb{Z}}$,
$\tilde{L}$ is diagonal embedding of L.

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times H & \xrightarrow{\pi_2} & H \\ \text{dense } \cup & & \cup & & \cup \text{ dense} \\ L & \overset{\text{bijective}}{\longleftrightarrow} & \tilde{L} & \overset{\text{bijective}}{\longleftrightarrow} & L^\star \end{array}$$

Overlap Automaton/Graph

Given: IFS $\Omega_i = \bigcup_{j=1}^{d} \lambda^\star \, \Omega_j + A_{ij}^\star$

Look at (ordered) intersection of pairs:

$$\Omega_i \cap x^\star + \Omega_{\hat{\imath}} = \bigcup_{j=1}^{d} \bigcup_{\hat{\jmath}=1}^{d} \bigcup_{t^\star \in A_{ij}^\star} \bigcup_{\hat{t}^\star \in A_{\hat{\imath}\hat{\jmath}}^\star} t^\star + \lambda^\star \left(\Omega_j \cap \left(\frac{x + \hat{t} - t}{\lambda} \right)^\star + \Omega_{\hat{\jmath}} \right)$$

Abbreviation: $\Xi_{(i\,,j\,,x)} = \Omega_i \cap x^\star + \Omega_j$.

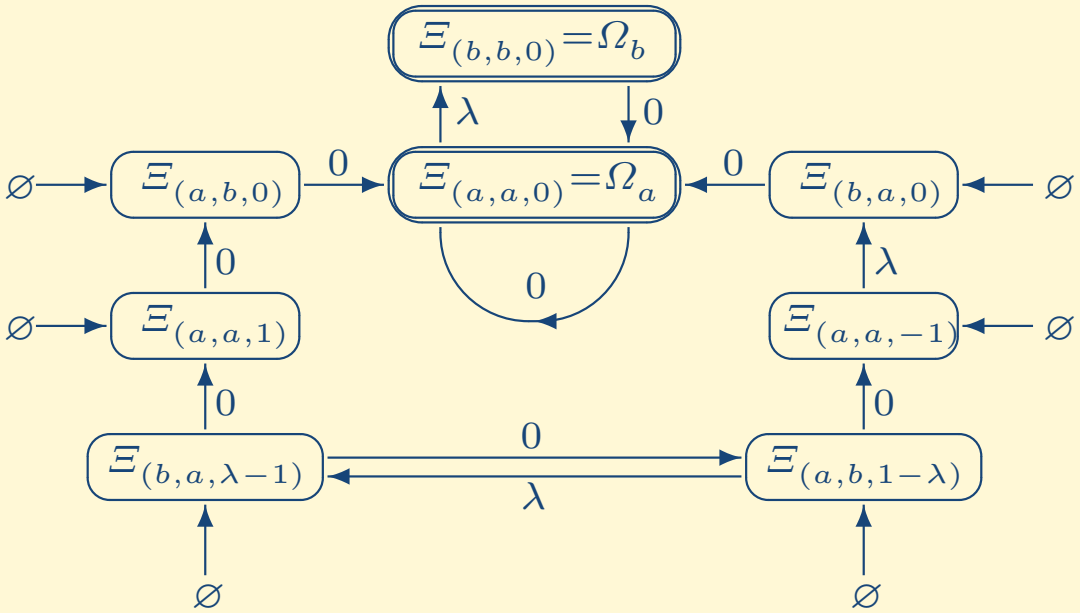
Start with vertices $\Xi_{(i\,,i\,,0)} = \Omega_i$, then obtain directed multigraph.

- If all $\Xi_{(i\,,j\,,0)}$ appear $\iff$ SCC
- If all $\Xi_{(i\,,j\,,x)}$ with $x^\star \in \Lambda(]-\ell_j\,,\ell_i[)$ appear $\iff$ GCC
- Remove all empty sets and the sets Ω_i $\implies$ IFS for $\bigcup_{i=1}^{d} \partial\Omega_i$

Another Automaton/Graph

Start with all sets $\Xi_{(i\,,j\,,x)}$ where $x^\star \in \Lambda(]-\ell_j\,,\ell_i[) \cap B_R(0)$.
Build directed multigraph as before.

- Same graph as before $\iff$ GCC
- Remove all empty sets and the sets Ω_i,
  calculate spectral radius σ of associated adjancency matrix
  - If GCC holds $\implies$ $\sigma < \lambda_1$
  - If covering degree > 1 (GCC fails) $\implies$ $\sigma \geq \lambda_1$

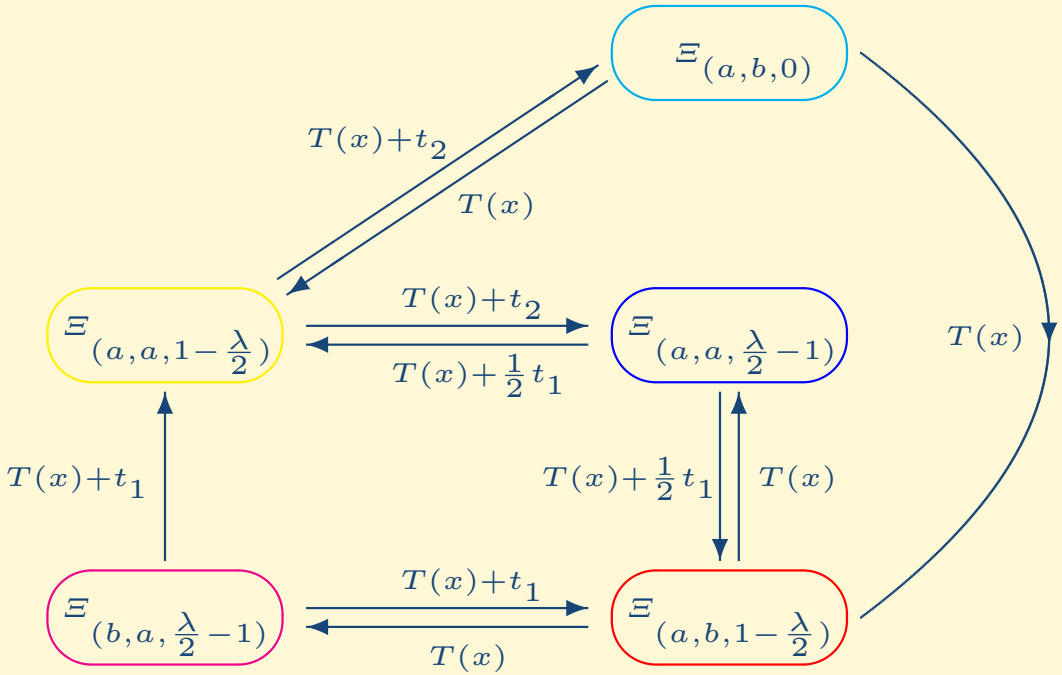


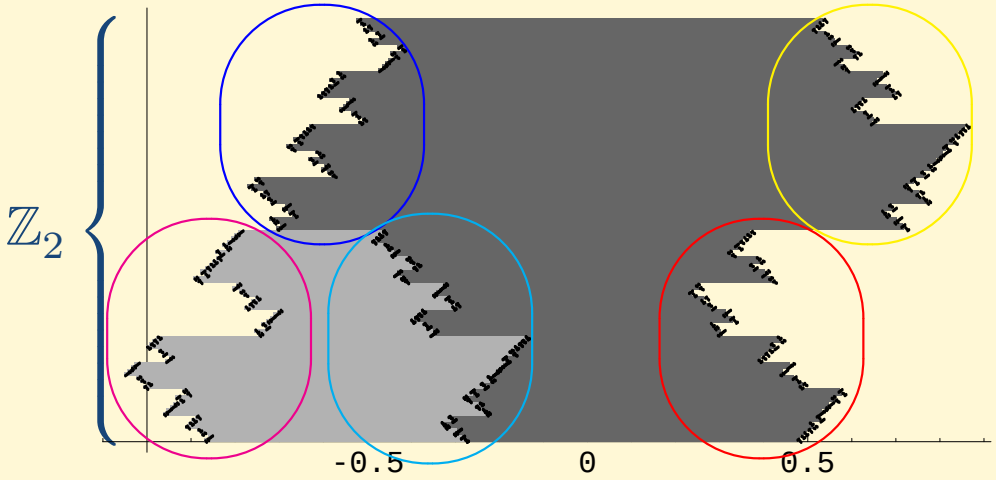
Conclusion

- Connection substitution matrix $\Leftrightarrow \lambda$.
- Dual tilings/coverings.
- Automata to decide GCC.
- Condition with spectral radius.

Substitution $a \mapsto aaba$, $b \mapsto aa$

$(a \mid a \mapsto aaba \mid aaba \mapsto \ldots aaaba \mid aabaa \ldots)$





$\dim_{\mathsf{Hd}} \Omega_a = \dim_{\mathsf{Hd}} \Omega_b = 2$

$1 \leq \dim_{\mathsf{Hd}}(\partial\Omega_a \cup \partial\Omega_b) \leq 1.1675$