

KOLAKOSKI- $(2m, 2n)$ ARE LIMIT-PERIODIC MODEL SETS

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ABSTRACT. We consider (generalized) Kolakoski sequences on an alphabet with two even numbers. They can be related to a primitive substitution rule of constant length ℓ . Using this connection, we prove that they have pure point dynamical and pure point diffractive spectrum, where we make use of the strong interplay between these two concepts. Since these sequences can then be described as model sets with ℓ -adic internal space, we add an approach to “visualize” such internal spaces.

1. INTRODUCTION

A one-sided infinite sequence ω over the alphabet $\mathcal{A} = \{1, 2\}$ is called a (classical) *Kolakoski sequence* (named after W. Kolakoski who introduced it in 1965, see [12]), if it equals the sequence defined by its run lengths, e.g.:

$$(1) \quad \omega = \underbrace{22}_2 \underbrace{11}_2 \underbrace{2}_1 \underbrace{1}_1 \underbrace{22}_2 \underbrace{1}_1 \underbrace{22}_2 \underbrace{11}_2 \underbrace{2}_1 \underbrace{11}_2 \dots = \omega.$$

Here, a *run* is a maximal subword consisting of identical letters. The sequence $\omega' = 1\omega$ is the only other sequence which has this property.

One way to obtain ω of (1) is by starting with 2 as a seed and iterating the two substitutions

$$\sigma_0 : \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 22 \end{array} \quad \text{and} \quad \sigma_1 : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 11, \end{array}$$

alternatingly, i.e., σ_0 substitutes letters on even positions and σ_1 letters on odd positions (we begin counting at 0):

$$2 \mapsto 22 \mapsto 2211 \mapsto 221121 \mapsto 221121221 \mapsto \dots$$

Clearly, the iterates converge to the Kolakoski sequence ω (in the obvious product topology), and ω is the unique (one-sided) fixed point of this iteration.

One can generalize this by choosing a different alphabet $\mathcal{A} = \{p, q\}$ (we are only looking at alphabets with $\text{card}(\mathcal{A}) = 2$). Such a (generalized) Kolakoski sequence, which is also equal to the sequence of its run lengths, can be obtained by iterating the two substitutions

$$(2) \quad \sigma_0 : \begin{array}{l} q \mapsto p^q \\ p \mapsto p^p \end{array} \quad \text{and} \quad \sigma_1 : \begin{array}{l} q \mapsto q^q \\ p \mapsto q^p \end{array}$$

alternatingly. Here, the starting letter of the sequence is p . We will call such a sequence a Kolakoski- (p, q) sequence, or $\text{Kol}(p, q)$ for short. The classical Kolakoski sequence ω of (1) is therefore denoted by $\text{Kol}(2, 1)$ (and ω' by $\text{Kol}(1, 2)$).

While little is known about the classical Kolakoski sequence (see [6]), and the same holds for all $\text{Kol}(p, q)$ with p odd and q even or vice versa (see [22]), the situation is more favorable if p and q are either both even or both odd. If both are odd, one can, in some cases, rewrite the substitution as a substitution of Pisot type (see [22, 3]), which can be described as (limit–)

aperiodic model sets. By this method, $\text{Kol}(3, 1)$ is studied in [3] and shown to be a deformed model set. The case where both symbols are even will be studied below.

It is the aim of this article to determine structure and order of the sequences $\text{Kol}(2m, 2n)$. This will require two steps: First we establish an equivalent substitution of constant length for $\text{Kol}(2m, 2n)$ and analyze it with methods known from the theory of dynamical systems. Then we conclude diffractive properties from this.

REMARK: Every $\text{Kol}(p, q)$ can uniquely be extended to a bi-infinite (or two-sided) sequence. The one-sided sequence (to the right) is $\text{Kol}(p, q)$ as explained above. The added part to the left is a reversed copy of $\text{Kol}(q, p)$, e.g., in the case of the classical Kolakoski sequence of (1), this reads as

$$\dots 11221221211221|22112122122112\dots,$$

where “|” denotes the seamline between the one-sided sequences. Note that, if $q = 1$ (or $p = 1$), the bi-infinite sequence is mirror symmetric around the first position to the left (right) of the seamline. The bi-infinite sequence equals the sequence of its run lengths, if counting is begun at the seamline. Alternatively, one can get such a bi-infinite sequence by starting with $q|p$ and applying the two substitutions to get $\sigma_1(q)|\sigma_0(p)$ in the first step and so forth. This also implies that $\text{Kol}(p, q)$ and $\text{Kol}(q, p)$ will have the same spectral properties, and it suffices to study one of them.

2. $\text{KOL}(2m, 2n)$ AS SUBSTITUTION OF CONSTANT LENGTH

If both letters are even numbers, i.e., $p = 2m$ and $q = 2n$ (with $m \neq n$, where we can concentrate on $m > n$ by the above discussion), one can build blocks of two letters and obtain an (ordinary) substitution. Setting $A = pp$ and $B = qq$, these substitutions and their *substitution matrix* \mathbf{M} (sometimes called *incidence matrix* of the substitution) are given by

$$(3) \quad \sigma: \begin{array}{l} A \mapsto A^m B^m \\ B \mapsto A^n B^n \end{array} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} m & m \\ n & n \end{pmatrix},$$

where the entry M_{ij} is the number of occurrences of j in $\sigma(i)$ ($i, j \in \{A, B\}$; sometimes the transposed matrix is used). A bi-infinite fixed point can be obtained as follows:

$$B|A \mapsto A^n B^n | A^m B^m \mapsto \dots$$

This corresponds to the unique bi-infinite $\text{Kol}(2m, 2n)$ according to our above convention.

A substitution ϱ is *primitive* if the corresponding substitution matrix \mathbf{M} is primitive, i.e., \mathbf{M}^k has positive entries only for some $k \in \mathbb{N}$. Equivalently, ϱ is primitive if there exists a positive integer $k \in \mathbb{N}$ such that every $i \in \mathcal{A}$ occurs in $\varrho^k(j)$ for all $j \in \mathcal{A}$. The vector ℓ with components $\ell_i = |\varrho(i)|$, for $i \in \mathcal{A}$, is called the *length* of the substitution ϱ . If all ℓ_i are equal, ϱ is a substitution of *constant length*. For the substitution σ of (3), we have

$$\ell = \begin{pmatrix} 2m \\ 2n \end{pmatrix},$$

which is therefore not of constant length (recall that $m \neq n$).

We will also need some notions from the theory of *dynamical systems*, see [18] and [8, Chapters 1, 5 and 7] for details. Let ϱ be a primitive substitution over \mathcal{A} and $u \in \mathcal{A}^{\mathbb{Z}}$ a bi-infinite fixed point of ϱ (i.e., $u = \varrho^k(u)$ for some $k \in \mathbb{N}$). Denote by u_k the k th letter of u

($k \in \mathbb{Z}$) and by T the shift map (i.e., $(T(u))_k = u_{k+1}$). Let \mathcal{A} be equipped with the discrete and $\mathcal{A}^{\mathbb{Z}}$ with the corresponding product topology. If we set

$$X(\varrho) = \overline{\{T^k(u) \mid k \in \mathbb{Z}\}},$$

then $(X(\varrho), T)$ is a *dynamical system*. Since we require ϱ to be primitive, this dynamical system is *minimal* (i.e., $\{T^k(u) \mid k \in \mathbb{Z}\}$ is dense in $X(\varrho)$ for all $u \in (X(\varrho), T)$), does not depend on the chosen fixed point u (if more than one exists, which is possible in the two-sided situation) and has a unique probability measure μ associated with it. In other words, it is *strictly ergodic*. On the Hilbert space $\mathcal{L}^2(X(\varrho), \mu)$, we have the unitary operator

$$\begin{aligned} U : \mathcal{L}^2(X(\varrho), \mu) &\rightarrow \mathcal{L}^2(X(\varrho), \mu), \\ f &\mapsto f \circ T. \end{aligned}$$

If $Uf = e^{i\lambda}f$ for some $0 \neq f \in \mathcal{L}^2(X(\varrho), \mu)$, we call $e^{i\lambda}$ an *eigenvalue* of $(X(\varrho), T)$ and f the corresponding *eigenfunction*. The *spectrum* (of the dynamical system) is said to be a *pure point dynamical spectrum* (or *discrete spectrum*), if the eigenfunctions span $\mathcal{L}^2(X(\varrho), \mu)$. If 1 is the only eigenvalue and the only eigenfunctions are the constants, the spectrum is *continuous*. It is also possible that it has pure point and continuous components. In that case it is called *partially continuous*. Two dynamical systems (X, T) and (Y, S) are *isomorphic* (or *measure-theoretically isomorphic*), if there exists an invertible measurable map $\varphi : X \rightarrow Y$, almost everywhere defined, such that φ preserves the measure and the dynamics (i.e., $\varphi \circ T = S \circ \varphi$).

The spectral theory of primitive substitutions of constant length is well understood. By the following criterion, we know that the substitutions σ of (3) are related to substitutions of constant length.

Lemma 1. [5, Section V, Theorem 1] *Let ϱ be a substitution of nonconstant length ℓ . If ℓ is a right eigenvector of the corresponding substitution matrix \mathbf{M} , then $(X(\varrho), T)$ is isomorphic to a substitution dynamical system generated by a substitution of constant length. \square*

Since the substitutions σ of (3) fulfill¹ the requirements of this lemma, the next task is now to construct the corresponding substitutions of constant length. This is achieved by numbering the A 's and B 's in (3), i.e., we make the substitutions $A^m B^m \rightarrow A_1 \dots A_m B_1 \dots B_m$, respectively $A^n B^n \rightarrow A_{m+1} \dots A_{m+n} B_{m+1} \dots B_{m+n}$. Then, the former substitutions (3) induce

$$(4) \quad \begin{aligned} A_1 \dots A_m B_1 \dots B_m &\mapsto (A_1 \dots A_m B_1 \dots B_m)^m (A_{m+1} \dots A_{m+n} B_{m+1} \dots B_{m+n})^m \\ A_{m+1} \dots A_{m+n} B_{m+1} \dots B_{m+n} &\mapsto (A_1 \dots A_m B_1 \dots B_m)^n (A_{m+1} \dots A_{m+n} B_{m+1} \dots B_{m+n})^n. \end{aligned}$$

¹Note that from (2), one can also construct a primitive substitution by distinguishing odd and even positions, e.g., for Kol(4, 2) we would get (we use $\tilde{\cdot}$ as mark for even positions)

$$\begin{aligned} 4 &\rightarrow 4\tilde{4}\tilde{4}\tilde{4} \\ \tilde{4} &\rightarrow 2\tilde{2}2\tilde{2} \\ 2 &\rightarrow 4\tilde{4} \\ \tilde{2} &\rightarrow 2\tilde{2}. \end{aligned}$$

Instead of (3), we would get a substitution with substitution matrix

$$\mathbf{M} = \begin{pmatrix} m & m & 0 & 0 \\ 0 & 0 & m & m \\ n & n & 0 & 0 \\ 0 & 0 & n & n \end{pmatrix},$$

which is also primitive (\mathbf{M}^2 has positive entries only), but does not fulfill the requirements of Lemma 1. The eigenvalues of this \mathbf{M} are $\{0, 0, 0, m+n\}$.

From this we get substitutions of constant length $m + n$ (the eigenvalue of the substitution matrix \mathbf{M} in (3)) by parting the right sides in blocks of $m + n$ letters. For example, let $m = 2$ and $n = 1$. Then

$$\begin{array}{lcl} A_1 A_2 B_1 B_2 & \mapsto & A_1 A_2 B_1 \quad B_2 A_1 A_2 \quad B_1 B_2 A_3 \quad B_3 A_3 B_3 \\ A_3 B_3 & \mapsto & A_1 A_2 B_1 \quad B_2 A_3 B_3 \end{array}$$

and one extracts the following substitution of constant length 3:

$$(5) \quad \begin{array}{lcl} A_1 & \mapsto & A_1 A_2 B_1 \\ A_2 & \mapsto & B_2 A_1 A_2 \\ B_1 & \mapsto & B_1 B_2 A_3 \\ B_2 & \mapsto & B_3 A_3 B_3 \\ A_3 & \mapsto & A_1 A_2 B_1 \\ B_3 & \mapsto & B_2 A_3 B_3. \end{array}$$

In the same way, we get substitutions of constant length $m + n$ from (4). Note that these substitutions are all primitive since (compare to (4)) in every block of $2m^2 + 2n$ successive letters (note that we use $m > n$) every letter of $\mathcal{A} = \{A_1, \dots, A_{m+n}, B_1, \dots, B_{m+n}\}$ occurs, so if $(m + n)^{k_0} \geq 2m^2 + 2n$, then \mathbf{M}^{k_0} has positive entries only (this holds for $k_0 \geq 3$). Note also that we can reduce the alphabet by one letter by identifying $A_1 = A_{m+1}$ ($A_1 = A_3$ in the example (5)), because both A 's always yield the same substitution.

Let us now determine the positions of A_1 in the sequence u generated by (4). They are given by $\alpha \cdot 2m + \beta \cdot 2n$ for some $\alpha, \beta \in \mathbb{Z}$ (e.g., $0, 2m, 4m, \dots, 2m^2, 2m^2 + 2n, 2m^2 + 4n, \dots, 2m^2 + 2nm, \dots$). Therefore we get $\gcd\{i \mid u_i = u_0 = A_1\} = \gcd(2m, 2n) = 2 \gcd(m, n)$. The *height* $h(\varrho)$ of a primitive substitution ϱ of constant length ℓ which generates a sequence u is defined as

$$(6) \quad h(\varrho) = \max\{k \geq 1 \mid \gcd(k, \ell) = 1 \text{ and } k \text{ divides } \gcd\{i \mid u_i = u_0\}\}.$$

Then the following lemma holds.

Lemma 2. *Let $(X(\varrho), T)$ be a dynamical system, where ϱ is a primitive substitution of constant length ℓ and height $h(\varrho)$. Then the pure point part of this dynamical system is isomorphic to the dynamical system $(\mathbb{Z}_\ell \times \mathbb{Z}/h(\varrho)\mathbb{Z}, \tau)$, where τ is the addition of $(1, 1)$ on the Abelian group $\mathbb{Z}_\ell \times \mathbb{Z}/h(\varrho)\mathbb{Z}$, i.e., the direct product of the ℓ -adic integers \mathbb{Z}_ℓ and the cyclic group $\mathbb{Z}/h(\varrho)\mathbb{Z}$ of order $h(\varrho)$. Therefore the pure point dynamical spectrum is given by*

$$\left\{ e^{2\pi i \frac{n}{\ell m} + 2\pi i \frac{k}{h(\varrho)}} \mid k, n \in \mathbb{Z}, m \in \mathbb{N} \right\}.$$

Note that

$$(7) \quad \mathbb{Z}_\ell \simeq \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r},$$

where p_1, \dots, p_r are the distinct primes dividing ℓ , see [16, Section 3.10].

Proof. The lemma is just a reformulation of [5, Theorem II.13], compare with [18, Section VI.] and [8, Section 7.3]. \square

Proposition 1. *Suppose $(X(\sigma), T)$ has pure point dynamical spectrum, where σ is the substitution of (3). Then*

$$(X(\sigma), T) \simeq \begin{cases} (\mathbb{Z}_{m+n} \times \mathbb{Z}/2\mathbb{Z}, \tau) & \text{if } m+n \text{ is odd} \\ (\mathbb{Z}_{m+n}, \tilde{\tau}) & \text{if } m+n \text{ is even,} \end{cases}$$

where τ is the addition of $(1, 1)$ and $\tilde{\tau}$ the addition of 1.

Proof. For the substitution (4) of constant length $\ell = m + n$, we have already seen that $\gcd\{i \mid u_i = u_0\} = 2 \gcd(m, n)$. Therefore, using (6), the height of this substitution is 2 if ℓ is odd and 1 if ℓ is even. The dynamical system of the substitution (4) is isomorphic to $(X(\sigma), T)$ by Lemma 1, therefore they have the same spectrum. The remaining statement follows from Lemma 2. \square

We want to show that the spectrum of σ is indeed pure point. For this we use slightly different substitutions of constant length that we deduce from σ . We substitute $A^m B^m \rightarrow a_1 \dots a_m$ and $A^n B^n \rightarrow b_1 \dots b_n$ (so in (4) we build essentially blocks of two, e.g., $a_1 = A_1 A_2$). We get

$$(8) \quad \begin{aligned} a_1 \dots a_m &\mapsto (a_1 \dots a_m)^m (b_1 \dots b_n)^m \\ b_1 \dots b_n &\mapsto (a_1 \dots a_m)^n (b_1 \dots b_n)^n, \end{aligned}$$

which again gives substitutions of constant length $\ell = m + n$. They are all primitive substitutions by the same argument as before (in every block of $m^2 + n$ successive letters every letter occurs). In the case $n > 1$, we can reduce the alphabet by one letter by identifying $a_1 = b_1 \mapsto a_1 \dots a_m a_1 \dots a_n$. So we have two cases, $n = 1$ with substitutions

$$(9) \quad \tilde{\theta} : \begin{cases} a_1 &\mapsto a_1 & a_2 & a_3 & \dots & a_{m-1} & a_m & a_1 \\ a_2 &\mapsto a_2 & a_3 & a_4 & \dots & a_m & a_1 & a_2 \\ \vdots & & \dots & & & & & \\ a_{m-1} &\mapsto a_{m-1} & a_m & a_1 & \dots & a_{m-3} & a_{m-2} & a_{m-1} \\ a_m &\mapsto a_m & b_1 & b_1 & \dots & b_1 & b_1 & b_1 \\ b_1 &\mapsto a_1 & a_2 & a_3 & \dots & a_{m-1} & a_m & b_1 \end{cases}$$

and $n > 1$ with substitutions θ (it is cumbersome to write down such a θ in general form, but we will investigate its structure in the next section). Now the height of θ and $\tilde{\theta}$ is always 1, because if $n > 1$ we get $\gcd\{i \mid u_0 = u_i = a_1\} = \gcd(m, n)$, and if $n = 1$ we get $\gcd\{i \mid u_0 = u_i = a_1\} = \gcd(m, m + 1) = 1$.

Let ϱ be a primitive substitution of constant length ℓ and height $h(\varrho) = 1$. One says that ϱ admits a *coincidence*, if there exist a $k \in \mathbb{N}$ and $j < \ell^k$ such that $\varrho^k(i)_j$ is the same for all $i \in \mathcal{A}$ (the j th letter of each $\varrho^k(i)$ is the same, i.e., ϱ^k admits a column of identical values).

Lemma 3. [5, Section III, Theorem 7] *Let $(X(\varrho), T)$ be a substitution dynamical system of constant length and height $h(\varrho) = 1$. Then $(X(\varrho), T)$ has pure point dynamical spectrum if and only if ϱ admits a coincidence.* \square

If a substitution has height $h > 1$, one gets a substitution of height 1 by combining letters into blocks of h letters. If this new substitution has pure point dynamical spectrum, so has the original substitution of height h , see [5]. Obviously, we get the following: if the substitutions θ and $\tilde{\theta}$ (which arise from (8)) admit coincidences, then the dynamical systems defined by σ of (3) have pure point dynamical spectrum.

3. COINCIDENCES AND COINCIDENCE MATRIX

Let us first check $\tilde{\theta}$ of (9) for coincidences. For this we begin by exploring the structure: $\tilde{\theta}(a_1)$ has two a_1 's at position 0 and m , $\tilde{\theta}(a_2)$ has an a_1 at position $m - 1$, etc. We get an a_1 in $\tilde{\theta}(a_k)$ at position $m + 1 - k$ for $1 \leq k \leq m - 1$. Similar arguments show that there is an a_m in $\tilde{\theta}(a_k)$ at position $m - k$ for $1 \leq k \leq m$ and at $m - 1$ in $\tilde{\theta}(b_1)$. Now, $\tilde{\theta}(a_m)$ has b_1 's at all

positions $1, \dots, m$. Furthermore, $\tilde{\theta}(b_1)$ has a b_1 at position m and shares the first m letters with $\tilde{\theta}(a_1)$. Schematically, we get the following structure of $\tilde{\theta}$:

$$(10) \quad \begin{array}{rcccccccc} a_1 & \mapsto & * & * & * & \dots & * & a_m & a_1 \\ a_2 & \mapsto & . & . & . & \dots & a_m & a_1 & . \\ \vdots & & & & & & & & \\ \vdots & & & & & / & / & & \\ a_{m-1} & \mapsto & . & a_m & a_1 & \dots & . & . & . \\ a_m & \mapsto & a_m & b_1 & b_1 & \dots & b_1 & b_1 & b_1 \\ b_1 & \mapsto & * & * & * & \dots & * & a_m & b_1 \end{array}$$

Here we have omitted (.) all letters that are not necessary and by * we denote the part that $\tilde{\theta}(a_1)$ and $\tilde{\theta}(b_1)$ share. We now check for *pairwise coincidences*, i.e., for $i_1, i_2 \in \mathcal{A}$ we check whether there is a $k \in \mathbb{N}$ and a $j < \ell^k = (m+1)^k$ such that $\sigma^k(i_1)_j = \sigma^k(i_2)_j$.

So we pick $i_1, i_2 \in \mathcal{A} = \{a_1, \dots, a_m, b_1\}$, $i_1 \neq i_2$. Suppose $i_1 \neq a_m$ (otherwise we interchange i_1 and i_2). Then i_2 either equals a_m or at least $\tilde{\theta}(i_2)$ has an a_m (every $\tilde{\theta}(i)$, $i \in \mathcal{A}$ has one). In the first case take $k = 1$, otherwise $k = 2$. Observe that there are m successive b_1 's in $\tilde{\theta}(a_m)$. So, if we look at $\tilde{\theta}^k(i_1)$ and $\tilde{\theta}^k(i_2)$, we get the following: On the one hand, there are m successive b_1 's somewhere in $\tilde{\theta}^k(i_2)$, say at positions $j, \dots, j+m-1$. On the other hand, in $\tilde{\theta}^k(i_1)$, there is at one of these positions $j, \dots, j+m-1$ either a b_1 , and we have a pairwise coincidence, or an a_1 . Say there is an a_1 at \tilde{j} with $j \leq \tilde{j} \leq j+m-1$. Then in $\tilde{\theta}^{k+1}(i_1)$ and $\tilde{\theta}^{k+1}(i_2)$ we have pairwise coincidences at positions $\tilde{j} \cdot \ell, \dots, \tilde{j} \cdot \ell + m$ (the *'s of (10)).

From this pairwise coincidences we get a coincidence inductively: We start with two letters i_1, i_2 and after $k_1 \leq 3$ substitutions we have a pairwise coincidence, say at j_1 . Now a third letter i_3 may have something else at $\tilde{\theta}^{k_1}(i_3)_{j_1}$, but whatever it is, in $\tilde{\theta}^{k_1+k_2}$ ($k_2 \leq 3$) all three coincide somewhere at a position j_2 with $j_1 \cdot \ell^{k_2} \leq j_2 < (j_1 + 1) \cdot \ell^{k_2}$. Since there are $\text{card}(\mathcal{A}) = m+1$ letters, we get a coincidence after at most $3 \cdot m$ substitutions (i.e., there is a $j < \ell^{3m}$ such that all $\tilde{\theta}^{3m}(i)_j$ are the same for all $i \in \mathcal{A}$).

The structure of θ is different. We have $\mathcal{A} = \{a_1, \dots, a_m, b_2, \dots, b_n\}$ and therefore $\text{card}(\mathcal{A}) = m+n-1$. Let us first show an example, with $m = 5$ and $n = 3$:

$$(11) \quad \begin{array}{rcccccccc} a_1 & \mapsto & a_1 & a_2 & a_3 & a_4 & a_5 & a_1 & a_2 & a_3 \\ a_2 & \mapsto & a_4 & a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & \mapsto & a_2 & a_3 & a_4 & a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & \mapsto & a_5 & a_1 & b_2 & b_3 & a_1 & b_2 & b_3 & a_1 \\ a_5 & \mapsto & b_2 & b_3 & a_1 & b_2 & b_3 & a_1 & b_2 & b_3 \\ b_2 & \mapsto & a_4 & a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & a_1 \\ b_3 & \mapsto & b_2 & b_3 & a_1 & b_2 & b_3 & a_1 & b_2 & b_3 \end{array}$$

Since the positions of two consecutive a_1 's in the sequence differ by at most m , there is an a_1 in every $\theta(i)$ with $i \in \mathcal{A}$ (note that θ is a substitution of constant length $\ell = m+n$). Again we look for pairwise coincidences, so choose $i_1, i_2 \in \mathcal{A}$. Then there is (at least) one a_1 in $\theta(i_1)$, say at position j_1 , and (at least) one in $\theta(i_2)$, say at position j_2 . Since there can be more than one a_1 in either, we choose j_1, j_2 such that $|j_1 - j_2|$ is minimal. We further choose i_1, i_2 such that $j_1 < j_2$ (in the case $j_1 = j_2$, e.g., $i_1 = a_1$ and $i_2 = a_5$ in the above example, we are already done). If we look at $\theta(i_1)$ and $\theta(i_2)$, there are two cases each (and therefore four cases, if we look at the combinations): Either $\theta(i_1)_{j_1+1} = a_2, \dots, \theta(i_1)_{j_2} = a_{j_2+1-j_1}$ or

$\theta(i_1)_{j_1+1} = b_2, \dots, \theta(i_1)_{j_2} = b_{j_2+1-j_1}$ and either $\theta(i_2)_{j_1} = a_{m+1+j_1-j_2}, \dots, \theta(i_2)_{j_2-1} = a_m$ or $\theta(i_2)_{j_1} = b_{m+1+j_1-j_2}, \dots, \theta(i_2)_{j_2-1} = b_m$. This is all that can occur by the chosen minimality of $j_2 - j_1 > 0$.

Now we examine the case where $\theta(i_1)_{j_1+1} = a_2, \dots, \theta(i_1)_{j_2} = a_{j_2+1-j_1}$ and $\theta(i_2)_{j_1} = a_{m+1+j_1-j_2}, \dots, \theta(i_2)_{j_2-1} = a_m$. We want to show that $\theta^2(i_1)$ and $\theta^2(i_2)$ have a pairwise coincidence. Let us look at the a_1 's in $\theta(a_i)$ only (we use again the above example, but the reasons given apply for arbitrary m, n):

$$(12) \quad \begin{array}{cccccccc} a_1 & \mapsto & \overset{1}{\mathbf{a}}_1 & a_2 & a_3 & a_4 & a_5 & \overset{2}{\mathbf{a}}_1 & a_2 & a_3 \\ a_2 & \mapsto & a_4 & a_5 & \overset{3}{\mathbf{a}}_1 & a_2 & a_3 & a_4 & a_5 & \overset{4}{\mathbf{a}}_1 \\ a_3 & \mapsto & a_2 & a_3 & a_4 & a_5 & \overset{5}{\mathbf{a}}_1 & a_2 & a_3 & a_4 \\ a_4 & \mapsto & a_5 & \overset{6}{\mathbf{a}}_1 & b_2 & b_3 & \overset{7}{\mathbf{a}}_1 & b_2 & b_3 & \overset{8}{\mathbf{a}}_1 \\ a_5 & \mapsto & b_2 & b_3 & \overset{9}{\mathbf{a}}_1 & b_2 & b_3 & \overset{10}{\mathbf{a}}_1 & b_2 & b_3 \end{array}$$

First we number the a_1 's with $1, \dots, 2m$ (left to right in $\theta(a_i)$ and top ($i = 1$) to bottom ($i = m$)) and we will speak of the k th a_1 (with $1 \leq k \leq 2m$) according to that number. We observe the following:

- Let $k < m$. If the k th a_1 occurs at position $j \geq n$ in $\theta(a_i)$, then the $(k+1)$ -st a_1 occurs at position $j-n$ in $\theta(a_{i+1})$. If the k th a_1 occurs at position $j < n$, then the $(k+1)$ -st a_1 occurs at $j+m$ in the same $\theta(a_i)$.
- Let $k > m+1$. If the k th a_1 occurs at position $j \geq n$, then the $(k-1)$ -st a_1 occurs at $j-n$ in the same $\theta(a_i)$. If the k th a_1 occurs at position $j < n$ in $\theta(a_i)$, then the $(k-1)$ -st a_1 occurs at position $j+m$ in $\theta(a_{i-1})$.²
- The second and the $(2m)$ -th a_1 occur at the same position m in $\theta(a_1)$, respectively $\theta(a_m)$. With the previous two observations we get: the k th and the $(2m+2-k)$ -th a_1 occur at the same position for $1 < k < m$.
- The first and the $(m+1)$ -st a_1 occur in $\theta(a_i)$ where there is at least one more a_1 . This is obvious for the first a_i , for the $(m+1)$ -st observe that if it occurs at position $j < m$, then there is also one at $j+n$, and if it occurs at position $j \geq m$, then there is also one at $j-m$.

These observations are based on the facts that the length of the substitution is $m+n$ and that the position of the a_1 's in the sequence are separated by m or n only. Now the fact that the a_i always occur in ascending order (i.e., we have $a_1 a_2 a_3 \dots$ and not $a_3 a_1 a_2 \dots$ or something else) together with the first two observations essentially gives us an algorithm, which always yields a pairwise coincidence in $\theta^2(i_1)$ and $\theta^2(i_2)$. Let us explain it in our example (12): Suppose we have $i_1 = a_2$ and $i_2 = a_3$. Then we have $j_1 = 2$ and $j_2 = 4$. The first step is always to reduce j_2 by one, so we have $j'_2 = 3$. We have $j'_2 \neq j_1$, but there is a second a_1 in $\theta(a_m)$ (a_m occurs at position j'_2 in $i_2!$), so we can increment j_1 by 1 and get $j'_1 = 3$. We have $j'_1 = j'_2$, and $\theta(\theta(a_2)_{j'_1}) = \theta(a_2)$ and $\theta(\theta(a_3)_{j'_2}) = \theta(a_5)$ both have an a_1 at position 2 (the third respectively the ninth a_1). Therefore we get a pairwise coincidence in $\theta^2(a_2)$ and $\theta^2(a_3)$. This algorithm relies on the ‘‘contrary line break property’’.

²Notice the contrary behaviour of the first two observations in going to a different or staying in the same $\theta(a_i)$ and the position of the corresponding a_1 . We call this the ‘‘contrary line break property’’.

The other three cases are mutatis mutandis the same, see the positions of the a_1 's in (11). So, starting with any two $i_1, i_2 \in \mathcal{A}$ we get a pairwise coincidence in $\theta^2(i_1)$ and $\theta^2(i_2)$. Inductively like before, we get a coincidence after at most $2 \cdot (m + n - 2)$ substitutions. Therefore we have established the following.

Theorem 1. *($X(\sigma), T$) with σ of (3) has pure point dynamical spectrum. Also, the dynamical system of the substitutions of constant length as defined implicitly in (4) and $\theta, \tilde{\theta}$ of (8) and (9) have pure point dynamical spectrum. \square*

Proposition 1'. *We have*

$$(13) \quad (X(\sigma), T) \simeq \begin{cases} (\mathbb{Z}_{m+n} \times \mathbb{Z}/2\mathbb{Z}, \tau) & \text{if } m+n \text{ is odd,} \\ (\mathbb{Z}_{m+n}, \tilde{\tau}) & \text{if } m+n \text{ is even,} \end{cases}$$

where τ is the addition of $(1, 1)$ and $\tilde{\tau}$ the addition of 1 . \square

REMARK: Let ϱ be a primitive substitution of constant length ℓ and height 1 over the alphabet $\mathcal{A} = \{1, \dots, r\}$. Then we can define the coincidence matrix \mathbf{C} , which is a quadratic $\frac{1}{2}r \cdot (r+1) \times \frac{1}{2}r \cdot (r+1)$ matrix. The entries are defined as follows (where $t \leq s$ and $v \leq u$):

$$C_{(st)(uv)} = \begin{cases} |\{j \mid \varrho(s)_j = u \wedge \varrho(t)_j = u\}| & \text{if } u = v \\ |\{j \mid \varrho(s)_j = u \wedge \varrho(t)_j = v\}| + |\{j \mid \varrho(s)_j = v \wedge \varrho(t)_j = u\}| & \text{if } u \neq v \end{cases}$$

Note that the substitution matrix \mathbf{M} is a submatrix of \mathbf{C} , since $M_{su} = C_{(ss)(uu)}$. Also, \mathbf{C} has row sums ℓ . With this definition, Lemma 3 reads as follows.

Proposition 2. [18, Proposition X.1]³ *For $(X(\varrho), T)$ are equivalent:*

- (i) *$(X(\varrho), T)$ has pure point dynamical spectrum.*
- (ii) *ℓ is a simple eigenvalue of the corresponding coincidence matrix \mathbf{C} .* \square

Obviously, ℓ is an eigenvalue of \mathbf{C} (\mathbf{C}/ℓ is a stochastic matrix with row sum 1).

Now the above proof of Theorem 1 translates to the following statements for \mathbf{C} :

- For $\tilde{\theta}$, the third power of the coincidence matrix, \mathbf{C}^3 , has a column $(C_{(st)(a_1 a_1)}^3)$ with nonzero entries only.
- For θ , the square of the coincidence matrix, \mathbf{C}^2 , has a column $(C_{(st)(a_1 a_1)}^2)$ with nonzero entries only.

Lemma 4. [18, Lemma X.3] *Let \mathbf{B} be a quadratic matrix with nonnegative integral entries and row sums ℓ . If $B_{ij} \geq 1$ for all i and a fixed j , then ℓ is a simple eigenvalue of \mathbf{B} . \square*

³Note, however, that we use a definition of \mathbf{C} different from [18]. The coincidence matrix there has dimension $r^2 \times r^2$ and has the form (with the proper enumeration of the pairs)

$$\begin{pmatrix} M & 0 & 0 \\ R & P & Q \\ R & Q & P \end{pmatrix}^t,$$

while the one defined above has the form

$$\begin{pmatrix} M & 0 \\ R & P+Q \end{pmatrix}.$$

Here M, P, Q are quadratic matrices and M is the substitution matrix. The Proposition is true for both matrices, the proof is analogous.

This establishes the desired result that ℓ is a simple eigenvalue for the coincidence matrix \mathbf{C} of θ ($\tilde{\theta}$), since ℓ^2 (ℓ^3) is a simple eigenvalue of \mathbf{C}^2 (\mathbf{C}^3).

We end this section with an example. We take $\tilde{\theta}$ for $m = 2$, $n = 1$ and therefore the substitution

$$(14) \quad \begin{array}{l} a_1 \mapsto a_1 a_2 a_1 \\ a_2 \mapsto a_2 b_1 b_1 \\ b_1 \mapsto a_1 a_2 b_1. \end{array}$$

We get the coincidence matrix

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{C}^2 = \begin{pmatrix} 4 & 3 & 2 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 & 2 \end{pmatrix}.$$

Here, already \mathbf{C}^2 has columns with positive entries only. The eigenvalues of \mathbf{C} are $\{0, 0, 1, 1, 2, 3\}$.

4. MODEL SETS AND DIFFRACTION

A *model set* $\Lambda(\Omega)$ (or *cut-and-project set*) in *physical space* \mathbb{R}^d is defined within the following general cut-and-project scheme, see [17, 1],

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi} & \mathbb{R}^d \times H & \xrightarrow{\pi_{\text{int}}} & H \\ & \nearrow_{1-1} & \cup & \nearrow_{\text{dense}} & \\ & & \Gamma & & \end{array}$$

where the *internal space* H is a locally compact Abelian group, and $\Gamma \subset \mathbb{R}^d \times H$ is a *lattice*, i.e., a co-compact discrete subgroup of $\mathbb{R}^d \times H$. The projection $\pi_{\text{int}}(\Gamma)$ is assumed to be dense in internal space, and the projection π into physical space has to be one-to-one on Γ . The model set $\Lambda(\Omega)$ is

$$\Lambda(\Omega) = \{\pi(x) \mid x \in \Gamma, \pi_{\text{int}}(x) \in \Omega\} \subset \mathbb{R}^d,$$

where the *window* $\Omega \subset H$ is a relatively compact set with nonempty interior.

Let u be a bi-infinite sequence over $\mathcal{A} = \{1, \dots, r\}$ and $\nu : \mathcal{A} \rightarrow \mathbb{C}, i \mapsto c_i$ be a (bounded) function which assigns to every letter a complex number (the *scattering strength*). Then the *autocorrelation coefficients* $\eta(z)$ are given by

$$\eta(z) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \overline{\nu(u_n)} \cdot \nu(u_{n+z})$$

provided the limits exist. We write δ_z for the Dirac measure at z , i.e., $\delta_z(f) = f(z)$ for f continuous. Then the *correlation measure* γ of u is given by

$$\gamma = \sum_{z \in \mathbb{Z}} \eta(z) \delta_z,$$

and the *diffraction spectrum*⁴ is given by the Fourier transform $\hat{\gamma}$ of this measure. If $\hat{\gamma}$ is a sum of Dirac measures only, i.e., $\hat{\gamma} = \sum_{k \in S} d_k \cdot \delta_k$ with a countable set S (for any choice of complex numbers $(c_i)_{i \in \mathcal{A}}$), then u is *pure point diffractive*, i.e., the diffraction spectrum consists of *Bragg peaks* only. Also, the d_k 's are the square of the absolute value of the corresponding *Fourier-Bohr coefficient* at k and therefore non-negative (real) numbers. If there is no Dirac measure in $\hat{\gamma}$, except one at position 0, δ_0 , which is determined by the density of the structure only, then the diffraction spectrum is *continuous*.

For substitutive systems, the diffraction spectrum and the spectrum of the corresponding dynamical system are closely related, see [7, 19, 20, 13, 14].

Proposition 3. [13, Corollary 1.] *Let ϱ be a primitive substitution of constant length ℓ with height 1 over $\mathcal{A} = \{1, \dots, r\}$, where u is a fixed bi-infinite word of ϱ . Define $U_i = \{j \in \mathbb{Z} \mid u_j = i\}$ for all $i \in \mathcal{A}$. We have $\mathbb{Z} = U_1 \dot{\cup} \dots \dot{\cup} U_r$, where $\dot{\cup}$ denotes disjoint union. Then the following are equivalent:*

- (i) ϱ admits a coincidence.
- (ii) The U_i 's are model sets for

$$\begin{array}{ccccc}
 \mathbb{R} & \xleftarrow{\pi} & \mathbb{R} \times \mathbb{Z}_\ell & \xrightarrow{\pi_{\text{int}}} & \mathbb{Z}_\ell \\
 \cup & \swarrow_{1-1} & \cup & \nearrow_{\text{dense}} & \cup \\
 \mathbb{Z} & \xleftarrow{\quad} & \Gamma = \{(z, z) \mid z \in \mathbb{Z}\} & \longrightarrow & \mathbb{Z}
 \end{array}$$

- (iii) The sequence u and the sets⁵ U_i are pure point diffractive. □

Note that properties “one-to-one” and “dense” are obvious, the interesting part is that there exist relatively compact windows (with respect to the ℓ -adic topology) with nonempty(!) interior. We will discuss this point in the next section.

With this proposition, we know that θ and $\tilde{\theta}$ (as defined implicitly in (8)) generate sequences which are pure point diffractive. But we got every letter $i \in \mathcal{A}$ for the appropriate alphabet for θ , $\tilde{\theta}$ by building four-letter blocks in the substitution rule (2), e.g., in (14) we have $a_1 = 4444$, $a_2 = 2222$ and $b_1 = 4422$. Such a deterministic substitution rule (i.e., $\text{Kol}(2m, 2n)$ is *local derivable* from the sequence generated by θ , respectively $\tilde{\theta}$) does not change the nature of the diffraction spectrum, only the Fourier-Bohr coefficients. The diffraction spectrum of $\text{Kol}(2m, 2n)$ can be calculated from the one generated by θ , $\tilde{\theta}$ as follows: For $\text{Kol}(2m, 2n)$ we only have scattering strengths c'_{2m} and c'_{2n} . If we therefore choose c_i ($i \in \mathcal{A}$ with respect to θ , $\tilde{\theta}$) according to its four-letter-composition in $\{2m, 2n\}$, then the diffraction spectrum $\hat{\gamma}$ of θ , $\tilde{\theta}$ is also a diffraction spectrum of $\text{Kol}(2m, 2n)$, where $\text{Kol}(2m, 2n)$ is realized as an atom chain with atoms not on \mathbb{Z} but on $\frac{1}{4}\mathbb{Z}$ (we get the diffraction spectrum of $\text{Kol}(2m, 2n)$ realized on \mathbb{Z} by a simple rescaling with the factor 4). For the example (14),

⁴So we think of u as an atomic chain, where there is an atom of type u_n at position n with scattering strength $\nu(u_n)$. We represent this atom as $\nu(u_n) \cdot \delta_n$ and therefore get a (countable) sum of weighted Dirac measures with autocorrelation γ .

⁵By this we mean the special choice of ν , where we set $c_i = 1$ and $c_j = 0$ for all $j \neq i$.

this means that we choose

$$\begin{aligned} c_{a_1} &= c'_4 \cdot (1 + e^{-\frac{2\pi i}{4}} + e^{-2\frac{2\pi i}{4}} + e^{-3\frac{2\pi i}{4}}) = 0 \\ c_{a_2} &= c'_2 \cdot (1 + e^{-\frac{2\pi i}{4}} + e^{-2\frac{2\pi i}{4}} + e^{-3\frac{2\pi i}{4}}) = 0 \\ c_{b_1} &= c'_4 \cdot (1 + e^{-\frac{2\pi i}{4}}) + c'_2 \cdot (e^{-2\frac{2\pi i}{4}} + e^{-3\frac{2\pi i}{4}}) = (1 - i)(c'_4 - c'_2). \end{aligned}$$

So in this case, the diffraction spectrum of $\text{Kol}(4, 2)$ is given by the one of U_{b_1} only. All $\text{Kol}(2m, 2n)$ are pure point diffractive.

Lemma 5. *For a sequence $u = \phi(v)$, where v is a bi-infinite fixed point of a primitive substitution and $\phi : \mathcal{A}_v \rightarrow \mathcal{A}_u^*$ is a morphism (where u (v) is a sequence over \mathcal{A}_u (\mathcal{A}_v)), the following statements are equivalent:*

- (i) *The dynamical system of u has pure point dynamical spectrum.*
- (ii) *u has pure point diffraction spectrum.*

Proof. This is a (weak) conclusion of [14, Theorem 3.2]. □

We therefore obtain the following.

Theorem 2. *All $\text{Kol}(2m, 2n)$ have pure point diffraction and pure point dynamical spectrum.* □

REMARKS: The dynamical system of $\text{Kol}(2m, 2n)$ is isomorphic to

$$(15) \quad \begin{array}{ll} (\mathbb{Z}_{m+n} \times \mathbb{Z}/4\mathbb{Z}, \tau) & \text{if } m+n \equiv 1, 3 \quad (4) \\ (\mathbb{Z}_{m+n} \times \mathbb{Z}/2\mathbb{Z}, \tau) & \text{if } m+n \equiv 2 \quad (4) \\ (\mathbb{Z}_{m+n}, \tilde{\tau}) & \text{if } m+n \equiv 0 \quad (4), \end{array}$$

where τ is the addition of $(1, 1)$ and $\tilde{\tau}$ the addition of 1. This can be seen by the fact that we get $\text{Kol}(2m, 2n)$ from the one generated by σ of (3) by the substitution $A = pp$, $B = qq$, which corresponds just to doubling each letter in the latter one (see the substitution matrices in (3) and Footnote 1).

If we consider how to get from (4), respectively (8), to $\text{Kol}(2m, 2n)$ and compare this to Proposition 3, we get (compare with (15)): $\text{Kol}(2m, 2n)$, respectively U_{2m} , U_{2n} are model sets for

$$\begin{array}{ccc} \mathbb{R} & \xleftarrow{\pi} & \mathbb{R} \times \mathbb{Z}_{m+n} \times F & \xrightarrow{\pi_{\text{int}}} & \mathbb{Z}_{m+n} \times F \\ & \swarrow_{1-1} & \cup & \nearrow_{\text{dense}} & \\ & & \Gamma = \{(z, z, z \bmod \text{ord}(F)) \mid z \in \mathbb{Z}\} & & \end{array}$$

where

$$(16) \quad F \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } m+n \equiv 1, 3 \quad (4) \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m+n \equiv 2 \quad (4) \\ \{0\} & \text{if } m+n \equiv 0 \quad (4). \end{cases}$$

(As before, $\mathbb{Z}_{m+n} \simeq \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r}$, where p_1, \dots, p_r are the distinct primes dividing $m+n$.) The diffraction spectrum calculated from this cut-and-project scheme is consistent with the previous one, since the Fourier-Bohr coefficients which arise in each \mathbb{Z}_{m+n} separately are weighted by factors 1 , $e^{-\frac{2\pi i}{4}}$, $e^{-2\frac{2\pi i}{4}}$ or $e^{-3\frac{2\pi i}{4}}$ which depend on the element of the cyclic group

$\mathbb{Z}/4\mathbb{Z}$ (similar for the case $\mathbb{Z}/2\mathbb{Z}$), compare with [2] — but this is just how we calculated the c_i 's from c'_{2m} and c'_{2n} .

Let us show how one calculates the diffraction spectrum of U_{b_1} of (14) explicitly. This substitution can be written in recursive equations for U_{a_1} , U_{a_2} and U_{b_1} by observing at which position in which substitution a certain letter occurs (e.g., b_1 occurs in $\tilde{\theta}(a_2)$ at positions 1 and 2 and in $\tilde{\theta}(b_1)$ at position 2)⁶:

$$\begin{aligned} U_{a_1} &= 3U_{a_1} \cup 3U_{a_1} + 2 \cup 3U_{b_1} \\ U_{a_2} &= 3U_{a_1} + 1 \cup 3U_{a_2} \cup 3U_{b_1} + 1 \\ U_{b_1} &= 3U_{a_2} + 1 \cup 3U_{a_2} + 2 \cup 3U_{b_1} + 2, \end{aligned}$$

where $rU_i + s = \{r \cdot z + s \mid z \in U_i\}$. Iterating these equations and using $U_{a_1} \cup U_{a_2} \cup U_{b_1} = \mathbb{Z}$, one gets:

$$\begin{aligned} U_{b_1} &= (9\mathbb{Z} + 5) \cup (27\mathbb{Z} + 17) \cup (27\mathbb{Z} + 22) \cup (81\mathbb{Z} + 53) \\ &\cup (81\mathbb{Z} + 58) \cup (81\mathbb{Z} + 64) \cup (81\mathbb{Z} + 65) \cup \dots \end{aligned}$$

The Fourier transform of each lattice coset $\omega_{r\mathbb{Z}+s} = \sum_{z \in \mathbb{Z}} \delta_{r \cdot z + s}$ is easy to calculate:

$$\widehat{\omega_{r\mathbb{Z}+s}} = \frac{1}{r} e^{-2\pi i k s} \omega_{\mathbb{Z}/r}$$

Every Fourier-Bohr coefficients of $\widehat{U_{b_1}}$ are then given by the sum of the Fourier-Bohr coefficients of the corresponding $\widehat{\omega_{r\mathbb{Z}+s}}$. The structure of U_{b_1} (similar for all U_i that occur for substitutions of constant length) as a union of a countable but infinite set of (periodic) lattice cosets $r \cdot \mathbb{Z} + s$ gives rise to the name *limit-periodic*, see [9].

The *support* of the Bragg peaks of $\text{Kol}(2m, 2n)$ is given by

$$\begin{aligned} &\left\{ \frac{k}{4 \cdot (m+n)^s} \mid k \in \mathbb{Z}, s \in \mathbb{N}_0 \right\} \\ &= \left\{ \frac{k}{2^\varepsilon \cdot p_1^{s_1} \cdot \dots \cdot p_r^{s_r}} \mid k \in \mathbb{Z}, s_1, \dots, s_r \in \mathbb{N}_0, \varepsilon \in \{0, \dots, \log_2(\text{ord}(F))\} \right\}, \end{aligned}$$

where p_1, \dots, p_r are the distinct primes dividing $m+n$ and F is the cyclic group of (16). However, there need not be a Bragg peak on every point of the support, e.g., $\text{Kol}(8, 4)$ is equivalent to a substitution θ of constant length $\ell = m+n = 6$, but the positions of a letter a_i, b_i are separated by multiples of 2: The support in this case is better described by $\{\frac{k}{2^s} \mid k \in \mathbb{Z}, s \in \mathbb{N}_0\}$ than by $\{\frac{k}{2^s \cdot 3^r} \mid k \in \mathbb{Z}, s, r \in \mathbb{N}_0\}$.

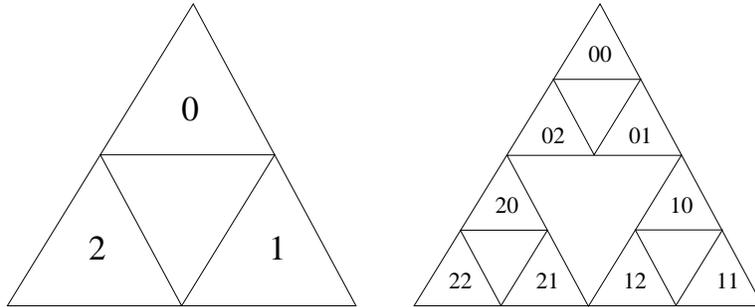
⁶Note that these recursive equations form an *iterated function system* (IFS) in 3-adic space, because multiplication by a factor of 3 is a contraction in the 3-adic topology. The closure of the windows in the 3-adic internal space is therefore given by the unique compact solution of this IFS by (a generalized version of) Hutchinson's theorem [11, Section 3.1(3)]. This method is well known for unimodular substitutions of Pisot-type, see [15], [3] and the vast literature about Rauzy fractals. Similar results apply for all primitive substitutions of constant length ℓ in ℓ -adic space.

5. EUCLIDEAN MODELS OF ℓ -ADIC INTERNAL SPACES

So far, we have talked in an “abstract” way about the ℓ -adic internal space. Usually the discussion ends at this point, but we want to “visualize” this ℓ -adic space. We hope that by doing this, we also gain some intuition for such spaces and the meaning of p -adic internal spaces for model sets.

Recall that a p -adic integer can be written as a formal series $t = \sum_{i \geq 0} t_i \cdot p^i$ with integral coefficients t_i satisfying $0 \leq t_i \leq p - 1$ (*Hensel expansion*). For the following, we identify a p -adic integer t with the sequence $(t_i)_{i \geq 0}$ of its coefficients. The set of all p -adic integers (a ring) is written as \mathbb{Z}_p , while the field of p -adic numbers is written as \mathbb{Q}_p and can be seen as the set of all Laurent series $\sum_{i \geq N} t_i \cdot p^i$ with $N \in \mathbb{Z}$. There is a p -adic valuation $v_p : \mathbb{Q}_p \setminus \{0\} \rightarrow \mathbb{Z}$ defined by $v_p(t) = \min\{i \in \mathbb{Z} \mid t_i \neq 0\}$, which gives rise to the p -adic metric with $|t|_p = p^{-v_p(t)}$ and $|0|_p = 0$ (and \mathbb{Q}_p and \mathbb{Z}_p are the completions of \mathbb{Q} and \mathbb{Z} with respect to the p -adic metric). So with respect to the p -adic metric, two numbers in \mathbb{Z} are close if their difference is divisible by a high power of p . Note that this is a *non-Archimedean* absolute value (i.e., $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}_p$) and we therefore get some “strange” properties: all triangles are isosceles, every point inside a ball $B_r(x) = \{y \mid |y - x|_p < r\}$ is the center of this ball, all balls are open and closed, etc., see [10].

For Euclidean models (see [21, Section 1.2]) of \mathbb{Z}_p we only need to know the formal series $t = \sum_{i \geq 0} t_i \cdot p^i$. We can even show models for \mathbb{Z}_ℓ , where we do not make use of (7). For this, we use the addressing scheme known for fractals, for example in the Sierpinsky gasket, see [4, Chapter IV.]:



Now the interesting thing here is that each point in the Sierpinsky gasket has a unique address (at least if we do not take the usual connected Sierpinsky gasket but the totally disconnected version; this can be obtained by using a contraction factor less than $\frac{1}{3}$ in the IFS for the Sierpinsky gasket). So each point in the Sierpinsky gasket corresponds to a sequence $(t_i)_{i \geq 0}$ with elements $0 \leq t_i \leq 2$ — this is just the Hensel expansion of the 3-adic integers. Similarly, the Cantor set is such a geometric encoding of the 2-adic integers. “Reasonable” geometric representations of \mathbb{Z}_ℓ in \mathbb{R}^d are those, where the sets $K_{\{x_0 \dots x_r\}} = \{t \in \mathbb{Z}_\ell \mid t_0 = x_0, \dots, t_r = x_r\}$ of points starting with the same address are represented by objects of the same size for a fixed $r \in \mathbb{N}$. Therefore we get that in d -dimension, \mathbb{Z}_ℓ with $d + 1 \leq \ell \leq (\text{kissing number in } \mathbb{R}^d) + 1$ can reasonably be represented, if we do not make use of (7). Note that we can represent \mathbb{Z}_3 either in \mathbb{R}^2 or \mathbb{R} .

This geometric representation surely fails for some p -adic (or ℓ -adic) properties (all triangles are isosceles, every point inside a ball is its center, etc.), but some are also “preserved”: points which are close in the p -adic topology are also close in this geometric representation and the

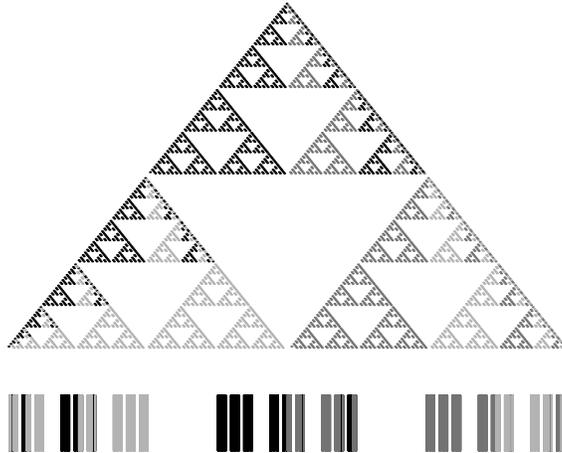


FIGURE 1. 3-adic model for the internal space of (14) in \mathbb{R}^2 (above) and \mathbb{R} (below, stretched for better representation). The colors correspond to a_1 (black), a_2 (dark gray) and b_1 (light gray).

representation as totally disconnected fractal corresponds to the totally disconnected field \mathbb{Q}_p , \mathbb{Z}_p and its geometric models are both compact sets. And balls in the p -adic topology correspond to scaled down copies of the whole fractal.

We like to conclude this section with our example from (14). The 3-adic geometric models are given in Figure 1. Observe that, in the two-dimensional representation, the parts (according to our above addressing scheme for the Sierpinsky gasket) $K_{\{02\}}$, $K_{\{12\}}$ and $K_{\{21\}}$ are colored by only one color. This corresponds to the fact that at positions $9\mathbb{Z} + 6$ are a_1 's, on $9\mathbb{Z} + 7$ are a_2 's and on $9\mathbb{Z} + 5$ are b_1 's only in the bi-infinite sequence. So, large patches of the same color in the geometric representations correspond to lattice cosets $\ell^r \mathbb{Z} + s$ with small r . A similar addressing scheme can be used for the one-dimensional representation (and in fact for all ℓ -adic representations).

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REFERENCES

- [1] M. Baake, "A guide to mathematical quasicrystals", in *Quasicrystals*, edited by J.-B. Suck, M. Schreiber and P. Häussler, Springer, Berlin, 2002, pp. 17–48; math-ph/9901014.
- [2] M. Baake, P. Kramer, M. Schlottmann and D. Zeidler, "Planar patterns with fivefold symmetry as sections of periodic structures in 4-space", *Int. J. Mod. Phys. B* **4** (1990), 2217–2268.
- [3] M. Baake and B. Sing, "Kolakoski-(3, 1) is a (deformed) model set", preprint (2002); available at math.MG/020698.
- [4] M.F. Barnsley, "Fractals Everywhere", Academic, Boston, 1988.
- [5] F.M. Dekking, "The spectrum of dynamical systems arising from substitutions of constant length", *Z. Wahrscheinlichkeitstheorie verwandte Geb.* **41** (1978), 221–239.

- [6] F.M. Dekking, “What is the long range order in the Kolakoski sequence?”, in *The Mathematics of Long-Range Aperiodic Order*, edited by R.V. Moody, Kluwer, Dordrecht, 1997, pp. 115–125.
- [7] S. Dworkin, “Spectral theory and X-ray diffraction”, *J. Math. Phys.* **34** (1993), 2965–2967.
- [8] N.P. Fogg, “Substitutions in Dynamics, Arithmetics and Combinatorics”, *Lecture Notes in Mathematics* **1784**, edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel, Springer, 2002; available at <http://iml.univ-mrs.fr/editions/preprint00/book/prebookdac.html>.
- [9] F. Gähler and R. Klitzing, “The diffraction pattern of self-similar tilings”, in *The Mathematics of Long-Range Aperiodic Order*, edited by R.V. Moody, Kluwer, Dordrecht, 1997, pp. 141–174.
- [10] F.Q. Gouvêa, “ p -adic Numbers”, 2nd ed., Springer, Berlin, 1997.
- [11] J.E. Hutchinson, “Fractals and self-similarity”, *Indiana Univ. Math. J.* **30** (1981), 713–747.
- [12] W. Kolakoski, “Self generating runs, Problem 5304”, *Am. Math. Monthly* **72** (1965), 674.
- [13] J.-Y. Lee and R.V. Moody, “Lattice substitution systems and model sets”, *Discrete Comput. Geom.* **25** (2001), 173–201; math.MG/0002019.
- [14] J.-Y. Lee, R.V. Moody and B. Solomyak, “Pure point dynamical and diffraction spectra”, *Annales Henri Poincaré* **3** (2003), 1003–1018; mp_arc/02-39.
- [15] J.M. Luck, C. Godrèche, A. Janner and T. Janssen, “The nature of the atomic surfaces of quasiperiodic self-similar structures”, *J. Phys. A: Math. Gen.* **26** (1993), 1951–1999.
- [16] J.C. Martin, “Substitution minimal flows”, *Am. J. Math.* **93** (1971), 503–526.
- [17] R.V. Moody, “Model sets: a survey”, in *From Quasicrystals to More Complex Systems*, edited by F. Axel, F. Dénoyer and J.P. Gazeau, EDP Sciences, Les Ulis, and Springer, Berlin, 2000, pp. 145–166; math.MG/0002020.
- [18] M. Queffélec, “Substitution dynamical systems – Spectral analysis”, *Lecture Notes in Mathematics* **1294**, Springer, Berlin, 1987.
- [19] C. Radin and M. Wolff, “Space tilings and local isomorphism”, *Geom. Dedic.* **42** (1992), 355–360.
- [20] E.A. Robinson, Jr., “The dynamical theory of tilings and quasicrystallography”, in *Ergodic Theory of \mathbb{Z}^d -Actions*, edited by M. Pollicott and K. Schmidt, Cambridge U. P., Cambridge, 1996, pp. 451–473.
- [21] A.M. Robert, “A Course in p -adic Analysis”, *Graduate Texts in Mathematics* **198**, Springer, New York, 2000.
- [22] B. Sing, “Spektrale Eigenschaften der Kolakoski-Sequenzen”, Diploma-Thesis, Universität Tübingen, 2002, available from the author.

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