MATH2130: Ordinary Differential Equations

Exercise Sheet 7: Complex Numbers and Wronskian

Please hand solutions in at the lecture on Monday 15th March.

1.) Let $z, w \in \mathbb{C}$. Verify that

(a) $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$,

(b) $|z| = |\overline{z}|$, i.e., $\mathbb{C} \ni z \mapsto \overline{z} \in \mathbb{C}$ is an *isometry*,

(c) $|z \cdot w| = |z| \cdot |w|$, i.e., $|\,.\,| : (\mathbb{C}, \cdot) \to (\mathbb{R}, \cdot)$ is a *homomorphism*,

(d) $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.

Conclude that the *triangle inequality* $|z + w| \leq |z| + |w|$ holds.
Hint: compute $|z + w|^2$.

2.) (a) Show that for real numbers $m_1, m_2, \ldots, m_n$ one has

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ m_1 & m_2 & \cdots & m_n \\ m_1^2 & m_2^2 & \cdots & m_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{n-1} & m_2^{n-1} & \cdots & m_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (m_j - m_i)\,.$$

This special determinant is called *Vandermonde deteminant.*
(**Hint**: Induction in n.)

(b) Use the result in (a) to show that for pairwise distinct numbers $m_1, m_2, \ldots, m_n$, the functions

$$\mathrm{e}^{m_1 x}, \quad \mathrm{e}^{m_2 x}, \quad \ldots, \quad \mathrm{e}^{m_n x}$$

are linearly independent.