MATH2130: Ordinary Differential Equations

Exercise Sheet 11: Wronskians and Laplace transforms

Please hand solutions in at the lecture on Wednesday 14th April.

1.) Write the following differential equation as a system of differential equations of the first order. Hence, set up the (vector valued) Volterra equation (respectively, the Picard iteration).

$$y''(x) = 3\,x^2\,y'(x) - y^2(x) \qquad \text{with initial conditions } \ y(0) = 0 \text{ and } y'(0) = 1.$$

2.) Use Proposition IV.4.3 (respectively, its generalisation to n linearly independent solutions of a homogeneous linear differential equation of order n) to prove the following statement:
Let $y_1,\ y_2,\ \ldots,\ y_n$ be n linearly independent solutions of the homogeneous linear differential equation

$$y^{(n)} + f_{n-1}(x)\,y^{(n-1)} + \ldots + f_1(x)\,y' + f_0(x)\,y = 0,$$

and let

$$y_c = c_1\,y_1 + c_2\,y_2 + \ldots + c_n\,y_n$$

be its complementary function. Let y_p be a particular solution of the nonhomogeneous linear differential equation

$$y^{(n)} + f_{n-1}(x)\,y^{(n-1)} + \ldots + f_1(x)\,y' + f_0(x)\,y = q(x). \tag{1}$$

Then

$$y(x) = y_p(x) + y_c(x)$$

is a general solution of (1)*, i.e., every solution can be written in this way.*

3.) Show that $f(t) = \cosh(t)$ and $g(t) = \cosh^2(t)$ are of exponential order and compute their Laplace transforms.

4.) Consider the functions $t \mapsto f_n(t) = t^n$, $n \in \mathbb{N}$. Show that the function $t \mapsto f_1(t) = t$ has Laplace transform given by

$$\hat{f}_1(s) = \frac{1}{s^2}, \quad \forall s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0.$$

By induction, deduce that, for each $n \in \mathbb{N}$, f_n has Laplace transform given by

$$\hat{f}_n(s) = \frac{n!}{s^{n+1}}, \quad \forall s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 0.$$

Please turn over!

5.) Consider the functions

$$f(t) = \begin{cases} 0, & t \le 0 \\ 1, & 0 < t \le 1 \\ 0, & t > 1 \end{cases} \qquad g(t) = \begin{cases} 0, & t \le 0 \\ t, & 0 < t \le 1 \\ 2 - t, & 1 < t \le 2 \\ 0, & t > 2. \end{cases}$$

Hence $f(t) = H(t) - H(t-1)$ and $g(t) = tH(t) - 2(t-1)H(t-1) + (t-2)H(t-2)$, where H denotes the Heaviside function defined in lectures.

Compute the Laplace transforms $\hat{f} = \mathcal{L}\{f\}$ and $\hat{g} = \mathcal{L}\{g\}$. Verify that $\hat{g}(s) = (\hat{f}(s))^2$ (while, clearly, $g(t) \neq (f(t))^2$).