MATH2130: Ordinary Differential Equations

Exercise Sheet 10: Picard and Power Series

Please hand solutions in at the lecture on Wednesday 7th April.

1.) Show: If a function $f : D \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ where $D \subset \mathbb{R}^2$ is open has a continuous partial derivative f_y, then f satisfies a (local) Lipschitz condition with respect to y in D.
(**Hint:** Mean value theorem.)

2.) (a) Rewrite the initial value problem $\dot{x}(t) = t + x(t)$ with $x(0) = 1$ as integral equation ("*Volterra equation*").

(b) Determine recursively the first six Picard iterates x_0, x_1, x_2, x_3, x_4, x_5.

(c) Guess a formula for $x_n(t)$ and verify this formula by induction.

(d) Determine the limit $x(t) = \lim_{n\to\infty} x_n(t)$ and confirm that $x(t)$ is indeed a solution of this initial value problem.

(**Note:** Of course, we could have determined the solution $x(t)$ of this linear differential equation using variation of parameters, but we want to practise the Picard iteration here!)

3.) Picard's method can also be applied to a system of two (coupled) first order differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x,y) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(t,x,y).$$

In this case, the Picard iteration reads

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s), y_k(s))\ \mathrm{d}s \qquad \text{and} \qquad y_{k+1}(t) = y_0 + \int_{t_0}^{t} g(s, x_k(s), y_k(s))\ \mathrm{d}s.$$

Find the first four Picard iterates $x_0, y_0, x_1, y_1, x_2, y_2, x_3, y_3$ for the following initial value problems.

(a) $\frac{\mathrm{d}x}{\mathrm{d}t} = t + y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = t - x^2$, $x(0) = 2$ and $y(0) = 1$.

(b) $\frac{\mathrm{d}x}{\mathrm{d}t} = x \cdot t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = x - \mathrm{e}^t$, $x(0) = 1$ and $y(0) = -1$.

Please turn over!

4.) We consider the following differential equation of second order:

$$x'' - 2\,t\,x' - 2\,x \;=\; 0.$$

We would like to solve this equation by using power series.

(a) Find a recursive formula for the coefficients of the power series by using the method of undetermined coefficients.

(b) Which (well-known) power series solves this differential equation with initial conditions $x(0) = 1$ and $x'(0) = 0$.