MATH1130: Calculus II

V.2. Applications of Line Integrals of Vector Fields (Not examinable!)

Definition. A vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *conservative* if

$$\int_{C_1} \langle \boldsymbol{F}, \mathrm{d}\boldsymbol{r} \rangle = \int_{C_2} \langle \boldsymbol{F}, \mathrm{d}\boldsymbol{r} \rangle$$

for any two curves C_1 and C_2 in $\mathbb{R}^n$ connecting the same two points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$.

Definition. A vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is said to be *irrotational* if $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$.

Proposition V.2.2. *In view of the equation* $\operatorname{curl}(\operatorname{grad} f) = \boldsymbol{0}$, *the following proposition can be stated. Let* $\boldsymbol{F} : D \to \mathbb{R}^3$ *be a vector field (where* $D \subset \mathbb{R}^3$*). Then the following three statements are equivalent:*

- $\boldsymbol{F}$ *is irrotational;*
- *there exists a scalar field* $\phi : D \to \mathbb{R}$ *such that* $\boldsymbol{F} = \nabla\phi$ *(*ϕ *is called a* scalar potential *of* $\boldsymbol{F}$*);*
- $\boldsymbol{F}$ *is conservative (that is, the work integral is independent of the path).*

Calculating Scalar Potentials. Let $\boldsymbol{F} : D \to \mathbb{R}$ be a conservative vector field and let C be an arbitrary curve from a point $\boldsymbol{a} \in D \subset \mathbb{R}^3$ to $\boldsymbol{x}$. Then

$$\phi(\boldsymbol{x}) = \int_C \langle \boldsymbol{F}, \mathrm{d}\boldsymbol{r} \rangle$$

is a scalar potential for $\boldsymbol{F}$.

For work integrals along general curves the Fundamental Theorem of Calculus takes the following form. Let $\phi : D \to \mathbb{R}$ be a continuously differentiable scalar field and let C be a curve in D from $\boldsymbol{x}_0$ to $\boldsymbol{x}_e$. Then

$$\int_C \langle \nabla\phi, \mathrm{d}\boldsymbol{r} \rangle = \phi(\boldsymbol{x}_e) - \phi(\boldsymbol{x}_0).$$

Definition. A vector field $\boldsymbol{F} : D \to \mathbb{R}^3$ is called *solenoidal* (or *divergence-free*) if $\operatorname{div} \boldsymbol{F} = 0$.

Application to Electrostatics. Given an electric field $\boldsymbol{E}$ and no sources, Maxwell's equations state that

$$\operatorname{div} \boldsymbol{E} = 0, \quad \operatorname{curl} \boldsymbol{E} = \boldsymbol{0}.$$

Now

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{0} \iff \exists \phi : D \to \mathbb{R} \;\text{ such that } \boldsymbol{E} = \nabla\phi$$

and

$$\operatorname{div} \boldsymbol{E} = 0 \iff \langle \nabla, \nabla\phi \rangle = 0 \iff \nabla^2\phi = 0.$$

Therefore the potential ϕ of $\boldsymbol{E}$ satisfies Laplace's equation.

Please turn over!

Application to Particle Motion. Suppose $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a force field and that a particle of mass m moves along a curve C through this field. Further suppose that $\boldsymbol{r}(t)$ is the position of the particle at time $t \in [t_0, t_e]$ ($\boldsymbol{r}(t)$ is a parametric representation of C where the parameter is time). Recall that

- The particle velocity is $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{\mathrm{d}t}$;
- The kinetic energy is $K(t) = \frac{1}{2}m \cdot \|\boldsymbol{v}(t)\|^2$;
- The particle acceleration is $\boldsymbol{a}(t) = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$;
- Newton's Second Law states that

$$\boldsymbol{F}(\boldsymbol{r}(t)) = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}.$$

Therefore,

$$\begin{aligned}
\int_C \langle \boldsymbol{F}, \mathrm{d}\boldsymbol{r} \rangle &= \int_{t_0}^{t_e} \langle \boldsymbol{F}(\boldsymbol{r}(t)), \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \rangle \ \mathrm{d}t \\
&= \int_{t_0}^{t_e} m \, \langle \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t}, \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \rangle \ \mathrm{d}t \\
&= \frac{1}{2}m \int_{t_0}^{t_e} \frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \rangle \ \mathrm{d}t \\
&= \frac{1}{2}m \int_{t_0}^{t_e} \frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{v}(t), \boldsymbol{v}(t) \rangle \ \mathrm{d}t,
\end{aligned}$$

so using the Fundamental Theorem of Calculus yields

$$\int_C \langle \boldsymbol{F}, \mathrm{d}\boldsymbol{r} \rangle = \frac{1}{2}m \, \|\boldsymbol{v}(t_e)\|^2 - \frac{1}{2}m \, \|\boldsymbol{v}(t_0)\|^2 .$$

Hence the work done is the change in kinetic energy from time t_0 to time t_e – the conservation of energy principle.