THE UNIVERSITY OF THE WEST INDIES

EXAMINATIONS OF MARCH 2010

Code and Name of Course: MATH1130 – CALCULUS II

Date and Time: Monday, 29 March, 12:10–13:00 Duration: **50 MINUTES**

INSTRUCTIONS TO CANDIDATES: This paper has 3 pages and 3 questions.

No calculators may be brought in and used.
Full marks will be given for correct answers to TWO (2) questions.
Only the best two answers will contribute towards the assessment.

1. Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$ be a vector function.

   (a) What is meant by saying that $\boldsymbol{r}$ is differentiable at $t_0 \in \mathbb{R}$. [3]

   (b) Calculate the derivative of each of the following functions.

   (i) $\boldsymbol{r}(t) = (2\,\cos(2\,t), 3\,\sin(2\,t), 3\,t)$; [2]

   (ii) $\boldsymbol{r}(t) = \left(t^2\,\mathrm{e}^{-2\,t}, t^3\,\mathrm{e}^{-2\,t}, t^2 + 3\right)$. [3]

   (c) Evaluate the following integrals.

   (i) $\int_0^2 \boldsymbol{r}(t)\,\mathrm{d}t$ where $\boldsymbol{r}(t) = \left(t \cdot \mathrm{e}^t, (\sqrt{t})^3\right)$. [3]

   (ii) $\int_1^2 \boldsymbol{r}(t)\,\mathrm{d}t$ where $\boldsymbol{r}(t) = \left(\frac{1}{t}, 9\,t^2 \cdot \sqrt{t^3+1}\right)$. [4]

2. Consider the curve

$$\begin{aligned}\boldsymbol{r}(t) &= 3\,\sin\left(t^3+1\right)\,\boldsymbol{e}_1 + 3\,\cos\left(t^3+1\right)\,\boldsymbol{e}_2 - 4\,\left(t^3+1\right)\,\boldsymbol{e}_3 \\ &= \left(3\,\sin\left(t^3+1\right), 3\,\cos\left(t^3+1\right), -4\,\left(t^3+1\right)\right).\end{aligned}$$

   (a) For arbitrary t,

   (i) find the velocity; [2]

   (ii) find the speed; [2]

   (iii) find the unit tangent vector; [2]

   (iv) find the arc length measured from the point $\boldsymbol{r}(-1) = (0, 3, 0)$; [2]

   (v) find the principal unit normal vector; [3]

   (vi) find the curvature and verify that it is constant, i.e., it is independent of t. [2]

   (**Note**: Explicitly state how each of these quantities is calculated.)

   (b) What is the geometric interpretation of curvature? [2]

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth scalar function (i.e., f is differentiable with continuous derivative).

(a) Let $(x_0, y_0) \in \mathbb{R}^2$. What is meant by the following:

(i) the *partial derivatives* $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (x_0, y_0); [3]

(ii) the *gradient* of f at (x_0, y_0). [2]

(b) Using that for the directional derivative $\mathrm{D}_{\boldsymbol{a}} f(x_0, y_0)$ – i.e., for the directional derivative of f at (x_0, y_0) in the direction of $\boldsymbol{a} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ – we have that

$$\mathrm{D}_{\boldsymbol{a}} f(x_0, y_0) = \langle \nabla f(x_0, y_0), \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \rangle,$$

show that $\max_{\boldsymbol{a} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}} |\mathrm{D}_{\boldsymbol{a}} f(x_0, y_0)| = \|\nabla f(x_0, y_0)\|$. [3]

(c) Find the gradient of the function

(i) $f(x, y) = (x + y)^2 + y$. [2]

(ii) $f(x, y) = y \cdot \mathrm{e}^{\cos(x)}$. [2]

(d) Find the directional derivative of $f(x, y) = 5\, x\, y^2$ at $(1, 1)$ in the direction $(3, 4)$. [3]

Bonus Question:
State without proof a theorem which allows the classification of a critical point (x_0, y_0) of f (i.e., a point (x_0, y_0) such that $\nabla f(x_0, y_0) = (0, 0)$) as a local maximum, minimum or saddle point of f. [5]

END OF QUESTION PAPER