MATH1130: Calculus II

III.9. Spherical and Cylindrical Polar Coordinates – Volume Integrals (Not examinable!)

In some sense, spherical and cylindrical coordinates are the three-dimensional extensions of polar coordinates.

In *spherical polar coordinates*, any point $\boldsymbol{p}$ is considered as lying on a sphere of radius ρ centred at the origin $\mathbf{0} \in \mathbb{R}^3$. The point $\boldsymbol{p}$ may be thought of as a point on the surface of the Earth (of radius ρ), where the *polar* angle, θ, is $\pi/2-$latitude and the *azimuthal* angle, ϕ, is longitude. Alternatively, consider $\boldsymbol{p}$ as being defined by ordinary plane polar coordinates (ρ, θ) in a plane which has been rotated about the z-axis through angle ϕ from the plane $y = 0$. The following applet demonstrates this (however, note that the roles of θ and ρ are exchanged):
http://www.flashandmath.com/mathlets/multicalc/coords/shilmay23fin.html
The point $\boldsymbol{p}$ has the spherical polar coordinate (ρ, θ, ϕ). The change of variables from spherical to Euclidean coordinates in 3-space is given by

$$\boldsymbol{r}(\rho, \theta, \phi) = \rho \sin\theta \cos\phi \, \boldsymbol{e}_1 + \rho \sin\theta \sin\phi \, \boldsymbol{e}_2 + \rho \cos\theta \, \boldsymbol{e}_3,$$

where

$$(\rho, \theta, \phi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi).$$

So, explicitly, the transformation is given by

$$\begin{cases} x(\rho, \theta, \phi) = \rho \sin\theta \cos\phi; \\ y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi; \\ z(\rho, \theta, \phi) = \rho \cos\theta, \end{cases}$$

It is true that $x^2 + y^2 + z^2 = \rho^2$. In spherical polar coordinates, the gradient has the form

$$\nabla f = \frac{\partial f}{\partial \rho} \boldsymbol{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \boldsymbol{e}_\theta + \frac{1}{\rho \sin\theta} \frac{\partial f}{\partial \phi} \boldsymbol{e}_\phi.$$

In *cylindrical polar coordinates*, any point $\boldsymbol{p}$ is considered as lying on a circular cylinder of radius ρ with the z-axis as its axis. The plane containing the z-axis and $\boldsymbol{p}$ makes angle θ with the plane $y = 0$. Alternatively, consider $\boldsymbol{p}$ as being defined by ordinary plane polar coordinates (ρ, θ) in a plane lifted parallel to the xy-plane through distance z. In this case $\boldsymbol{p}$ has the cylindrical polar coordinates (ρ, θ, z). The change of variables from cylindrical to Euclidean coordinates in 3-space is given by

$$\boldsymbol{r}(\rho, \theta, z) = \rho \cos\theta \, \boldsymbol{e}_1 + \rho \sin\theta \, \boldsymbol{e}_2 + z \, \boldsymbol{e}_3,$$

where

$$(\rho, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}.$$

So, explicitly, the transformation is given by

$$\begin{cases} x(\rho, \theta, z) = \rho \sin\theta; \\ y(\rho, \theta, z) = \rho \sin\theta; \\ z(\rho, \theta, z) = z, \end{cases}$$

Please turn over!

It is true that $x^2+y^2=\rho^2$. In cylindrical polar coordinates, the gradient has the form

$$\nabla f = \frac{\partial f}{\partial \rho}\,\boldsymbol{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\,\boldsymbol{e}_z.$$

The volume integral of a scalar field $f : D \to \mathbb{R}$ over the region $D \subset \mathbb{R}^3$ parametrised by $\boldsymbol{r}(u,v,w)$ where $(u,v,w) \in D^*$ is defined as (compare change of variables formula):

$$\iiint_D f \;\mathrm{d}V = \iiint_{D^*} f(\boldsymbol{r}(u,v,w)) \left\| \langle \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}, \frac{\partial \boldsymbol{r}}{\partial w} \rangle \right\| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Example. The volume of the sphere with radius a in the first octant is calculated using spherical polar coordinates.

Then,

$$D = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x^2+y^2+z^2 = a^2, x, y, z \geq 0\}.$$

Solution. **Step 1** – Parameterise:

$$\boldsymbol{r}(\rho,\theta,\phi) = (\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta), \;\; (\rho,\theta,\phi) \in D^* = [0,a]\times[0,\pi/2]\times[0,\pi/2].$$

Step 2 – Calculate the volume element $\mathrm{d}V$:

$$\frac{\partial \boldsymbol{r}}{\partial \rho} \times \frac{\partial \boldsymbol{r}}{\partial \theta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \times \begin{pmatrix} \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi \\ -\rho\sin\theta \end{pmatrix} = \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} \mathrm{d}V &= \left\| \langle \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix}, \begin{pmatrix} -\rho\sin\theta\sin\phi \\ \rho\sin\theta\cos\phi \\ 0 \end{pmatrix} \rangle \right\| \mathrm{d}\rho\,\mathrm{d}\theta d\phi \\ &= \rho^2\sin\theta(\sin^2\theta+\cos^2\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta d\phi \\ &= \rho^2\sin\theta\,\mathrm{d}\rho\,\mathrm{d}\theta d\phi. \end{aligned}$$

Step 3 Evaluate the integral:

$$\begin{aligned} \iiint_D 1 \;\mathrm{d}V &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2\sin\theta \;\mathrm{d}\rho\,\mathrm{d}\theta d\phi \\ &= \int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{1}{3}\rho^3\sin\theta\right]_0^a \mathrm{d}\theta d\phi \\ &= \int_0^{\pi/2} \left[-\frac{1}{3}a^3\cos\theta\right]_0^{\pi/2} d\phi \\ &= \int_0^{\pi/2} \frac{1}{3}a^3 d\phi = \frac{1}{3}\left[\phi\right]_0^{\pi/2} = \frac{\pi a^3}{6}. \end{aligned}$$

Note that the procedure takes the form volume integral → triple integral → repeated integral. □