MATH1130: Calculus II

Exercise Sheet 8: Scalar functions, gradient and level sets

Please hand solutions in at the lecture on Tuesday 23rd March. Attempting Exercise 1 is worth 1% of the final mark.

1.) Let

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } x = 0 = y. \end{cases}$$

(a) Show that f is continuous at $\mathbf{0}$.

(b) Show that f is **not** differentiable at $\mathbf{0}$.
(**Hint**: Consider appropriate directions.)

2.) Find the gradient (and thus also the partial derivatives) of each of the following functions.

(a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

(b) $g(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

(c) $h(w,x,y,z) = \arctan(w + 4\,x + 3\,y + 5\,z)$

3.) For each of the following, find the directional derivative of f at the point $\boldsymbol{c}$ in the direction of the specified vector $\boldsymbol{w}$.

(a) $f(x,y) = 3\,x^2\,y$, $\boldsymbol{c} = (-2,1)$, $\boldsymbol{w} = (2,3)$

(b) $f(x,y) = 3\,x^2\,y$, $\boldsymbol{c} = (-2,1)$, $\boldsymbol{w} = (2,3)$, but using the definition of directional derivative directly.

(c) $f(x,y,z) = \ln(x^2 + 2\,y^2 + z^2)$, $\boldsymbol{c} = (2,1,1)$, $\boldsymbol{w} = (-1,2,3)$

(d) $f(w,x,y,z) = w\,x^2\,y\,z^2$, $\boldsymbol{c} = (2,1,-1,2)$, $\boldsymbol{w} = (1,-1,2,3)$

For the function in (d) also answer the following question: In which direction does f increase most rapidly at $\boldsymbol{c}$?

Please turn over!

4.) Let

$$f(x,y) = \begin{cases} \frac{x\,y\,(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } x = 0 = y. \end{cases}$$

(a) Show that $f_x(0,y) = -y$ for all y.

(b) Show that $f_y(x,0) = x$ for all x.

(c) Show that $f_{yx}(0,0) \neq f_{xy}(0,0)$.

(d) In view of Clairaut's theorem (Schwarz' lemma), what can be said about f?

5.) Find and classify all the critical points of $f(x,y) = 4\,x\,y - 2\,x^2 - y^4$.

6.) Using for example the applet at
http://www.flashandmath.com/mathlets/multicalc/contours/combo.html
sketch the level curves of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) $f(x,y) = \frac{x^2}{4} + \frac{y^2}{16}$

(b) $f(x,y) = (x-1)^2 + (y+3)^2$

(c) $f(x,y) = \sin(x\,y)$