# MATH1130: Calculus II

## Exercise Sheet 7: Motions in 3-space, curves in polar coordinates and continuous scalar fields

Please hand solutions in at the lecture on Tuesday 16th March.

1.) The motion of a particle in the plane is described using polar coordinates by $r = \mathrm{e}^{\theta}$ for $0 \leq \theta \leq 2\pi$.

   (a) Find the arc length of this spiral.

   (b) Show that the tangent vector to the curve makes a constant angle of $\frac{\pi}{4}$ with the position vector using polar coordinates.

2.) Consider the curve $\boldsymbol{r}(t) = (\mathrm{e}^t, t, t^2)$.

   (a) Find the equation of the osculating plane at the point $t = 1$.

   (b) Find the equation of the osculating plane at the point $t = 0$.

   (**Hint**: Recall that the normal of the osculating plane is perpendicular to both the velocity and the acceleration vector.)

3.) Prove that if the speed $v = \|\boldsymbol{v}\| = \|\boldsymbol{r}'\|$ is a constant, then the acceleration is perpendicular to the velocity.

4.) Let $\boldsymbol{r}(t) = (\mathrm{e}^t, \mathrm{e}^{2\,t}, 1 - \mathrm{e}^t)$.

   (a) Write a parametric representation for the tangent line to $\boldsymbol{r}(t)$ at $t = 0$.

   (b) Find the equation of the osculating circle to $\boldsymbol{r}(t)$ at $t = 0$.

   (c) The curve $\boldsymbol{r}(t)$ intersects the curve $\boldsymbol{\ell}(s) = (1 - s, \cos(s), \sin(s))$ at the point $(1, 1, 0)$. What is the angle between their tangents at that point?

*Please turn over!*

5.) Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + 4y^2} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } x = 0 = y. \end{cases}$$

(a) Define $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ by $\boldsymbol{\alpha}(t) = (t, 0)$. Show that $\lim_{t\to 0} f(\boldsymbol{\alpha}(t)) = 0$.

(b) Define $\boldsymbol{\beta} : \mathbb{R} \to \mathbb{R}^2$ by $\boldsymbol{\beta}(t) = (0, t)$. Show that $\lim_{t\to 0} f(\boldsymbol{\beta}(t)) = 0$.

(c) Show that for any real number $m$, if we define $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ by $\boldsymbol{\gamma}(t) = (t, m \cdot t)$, then $\lim_{t\to 0} f(\boldsymbol{\gamma}(t)) = 0$.

(d) Define $\boldsymbol{\delta} : \mathbb{R} \to \mathbb{R}^2$ by $\boldsymbol{\delta}(t) = (t, t^2)$. Show that $\lim_{t\to 0} f(\boldsymbol{\delta}(t)) = \frac{1}{5}$.

(e) What can be concluded about $\lim_{(x,y)\to(0,0)} f(x,y)$?

(f) Plot the graph of $f$ and explain your results in terms of the graph.