MATH1130: Calculus II

EXERCISE SHEET 6: LENGTH OF CURVES, VELOCITY, SPEED AND ACCELERATION

Please hand solutions in at the lecture on Tuesday 9th March. Attempting Exercise 1 is worth 1% of the final mark.

1*.) (a) Suppose a particle moves along a curve C_1 in $\mathbb{R}^3$ so that its postition at time t is given by $\boldsymbol{r}(t)$. Let $\boldsymbol{v}$, v and $\boldsymbol{a}$ denote the velocity, speed and acceleration of the particle, respectively, and let κ be the curvature of C_1.

Using the facts $\boldsymbol{v} = v\,\hat{\boldsymbol{T}}$ and $\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\,\hat{\boldsymbol{T}} + v^2\,\kappa\,\hat{\boldsymbol{N}}$, show that

$$\boldsymbol{v} \times \boldsymbol{a} \;=\; v^3\,\kappa\,\left(\hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}}\right).$$

(b) Use the result of part (a) to show that

$$\kappa \;=\; \frac{\|\boldsymbol{v} \times \boldsymbol{a}\|}{\|\boldsymbol{v}\|^3}.$$

(c) Now let C_2 be the curve in $\mathbb{R}^2$ which is the graph of the function $f : \mathbb{R} \to \mathbb{R}$, i.e., C_2 is parametrized by $\tilde{\boldsymbol{r}}(t) = (t, f(t))$. Use the result from (b) to show that the curvature of C_2 at the point $\tilde{\boldsymbol{r}}(t)$ is

$$\kappa = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{\frac{3}{2}}}.$$

(d) Let $\tilde{C}$ be the graph of $g(t) = t^3$. Use the result from (c) to find the curvature of $\tilde{C}$ at $(1,1)$ and $(2,8)$. Give a geometric interpretation of these curvatures.

2.) The curve parametrized by

$$\boldsymbol{r}(t) = (\sin(2\,t)\,\cos(t), \sin(2\,t)\,\sin(t))$$

has four "petals". Find the length of one of these petals.
Note: If you are not able to evaluate the integral you are getting here, try to find an approximate numerical value (with justification).

3.) The position of an object at time t is given by $\boldsymbol{r}(t) = (t, t^2 - 2, 2\,t)$. Find the velocity, the speed, and the tangential and normal components of acceleration.