MATH1130: Calculus II

Exercise Sheet 1: Vectors in Euclidean space

Please hand solutions in at the lecture on Tuesday 2nd February.

1.) Let $\boldsymbol{x} = (1, 2)$, $\boldsymbol{y} = (2, 3)$ and $\boldsymbol{z} = (-2, 4)$. For each of the following, compute the indicated point $\boldsymbol{w}$ and its norm. Also, plot the points involved.

(a) $\boldsymbol{w} = \boldsymbol{x} + \boldsymbol{y}$

(b) $\boldsymbol{w} = \boldsymbol{z} - 2\,\boldsymbol{x}$

(c) $\boldsymbol{w} = 3\,\boldsymbol{x} + 2\,\boldsymbol{y} + \boldsymbol{z}$

2.) Calculate the distance between and scalar product of the following pairs of vectors. Also, find the angle between them.

(a) $\boldsymbol{x} = (3, 2)$, $\boldsymbol{y} = (-1, 3)$

(b) $\boldsymbol{x} = (3, -3, 0)$, $\boldsymbol{y} = (-1, 2, -5)$

(c) $\boldsymbol{x} = (1, 2, 4, -2, 3, -1)$, $\boldsymbol{y} = (3, 2, 1, -3, 2, 1)$

3.) For each of the following, find the component of $\boldsymbol{x}$ along $\boldsymbol{y}$ and the projection $\boldsymbol{w}$ of $\boldsymbol{x}$ onto $\boldsymbol{y}$. In each case verify that $\boldsymbol{y}$ is perpendicular to $(\boldsymbol{x} - \boldsymbol{w})$.

(a) $\boldsymbol{x} = (-2, 4)$, $\boldsymbol{y} = (4, 1)$

(b) $\boldsymbol{x} = (4, 1, 4)$, $\boldsymbol{y} = (-1, 3, 1)$

(c) $\boldsymbol{x} = (1, 2, 4, -1)$, $\boldsymbol{y} = (2, -1, 2, 3)$

Please turn over!

4.) Verify properties (ii) and (iii) in Proposition I.2.1 for the scalar product $\langle \cdot, \cdot \rangle$, i.e., show:

(a) If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are three vectors in $\mathbb{R}^n$, then

$$\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle \;=\; \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle \;=\; \langle \boldsymbol{y} + \boldsymbol{z}, \boldsymbol{x} \rangle.$$

(b) For all $c \in \mathbb{R}$ we have

$$\langle c \cdot \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; c \cdot \langle \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; \langle \boldsymbol{x}, c \cdot \boldsymbol{y} \rangle.$$

Make sure that you justified each step!

5.) Let $\boldsymbol{u} = (1, 1)$ and $\boldsymbol{v} = (-1, 2)$ be vectors in $\mathbb{R}^2$.

(a) Let $\boldsymbol{x} = (2, 1)$. Find scalars a and b such that $\boldsymbol{x} = a\,\boldsymbol{u} + b\,\boldsymbol{v}$. Are a and b unique?

(b) Let $\boldsymbol{x} = (x, y)$ be an arbitrary vector in $\mathbb{R}^2$. Show that there exist unique scalars a and b such that $\boldsymbol{x} = a\,\boldsymbol{u} + b\,\boldsymbol{v}$.

The result in (b) shows that $\boldsymbol{u}$ and $\boldsymbol{v}$ form a *basis* for $\mathbb{R}^2$ which is different from the standard basis of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

(c) Show that the vectors $\boldsymbol{u} = (1, 1)$ and $\boldsymbol{w} = (-1, -1)$ do not form a basis for $\mathbb{R}^2$.
Hint: Show that there do not exist scalars a and b such that $\boldsymbol{x} = a\,\boldsymbol{u} + b\,\boldsymbol{w}$ when $\boldsymbol{x} = (2, 1)$.