MA30056: Complex Analysis

Exercise Sheet 9: Laurent series & Singularities

Please hand solutions in at the lecture on Monday 27th April.

1.) Let p be a non-constant polynomial. Show: $|p(z)| \to \infty$ as $|z| \to \infty$.

Optional and not examinable: Regarding p as a function from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\mathbb{C}$, what are we showing here in terms of singularities? Do other entire functions like exp or sin also have this property?

2.) Find the Laurent series expansions of $f(z) = \frac{1}{z(1-z)(2-z)}$ for the annuli

$$\text{(i)}\quad 0 < |z| < 1, \qquad \text{(ii)}\quad 1 < |z| < 2, \qquad \text{(iii)}\quad 2 < |z|.$$

Hint: Do not compute integrals.

3.) Find the Laurent series expansions of $f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$ for the annuli

$$\text{(i)}\quad 0 < |z| < 1, \qquad \text{(ii)}\quad 0 < |z-1| < 1, \qquad \text{(iii)}\quad 0 < |z-2| < 1.$$

Hint: Do not compute integrals.

4.) Let $f : D \to \mathbb{C}$ be holomorphic. We say that $z_0 \in D$ is a *zero of order* $m \in \mathbb{N}$ of f if the Taylor series expansion of f at z_0

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k, \qquad \text{where} \qquad a_m \neq 0.$$

Prove that $z_0 \in D$ is a zero of order $m \in \mathbb{N}$ iff there is a holomorphic function g with $g(z_0) \neq 0$ so that $f(z) = (z - z_0)^m g(z)$.

Conclude that the zeros of a nonzero holomorphic function are isolated.

Optional question:

5.) Convince yourself that $z \mapsto \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$, i.e., it is holomorphic in $\mathbb{C}$ except for poles.