MA30056: Complex Analysis

Exercise Sheet 9: Laurent series & Singularities

Please hand solutions in at the lecture on Monday 27th April.

1.) Let p be a non-constant polynomial. Show: $|p(z)| \to \infty$ as $|z| \to \infty$.

Optional and not examinable: Regarding p as a function from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\mathbb{C}$, what are we showing here in terms of singularities? Do other entire functions like exp or sin also have this property?

Solution: We have actually already shown this in the proof of Gauss' Theorem (see Exercise sheet 6 Question 4 or Exercise sheet 7 Question 1). Let $p(z) = \sum_{k=0}^{n} a_k z^k$ with $a_n \neq 0$. Then, since $\frac{p(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n} \to 1$ as $|z| \to \infty$, there is $R > 0$ so that

$$|z| > R \qquad \Rightarrow \qquad \left|\frac{p(z)}{a_n z^n}\right| \geq \frac{1}{2}.$$

Thus, $|p(z)| \geq \frac{1}{2}|a_n| \cdot |z|^n$ for $|z| > R$ and the claim follows.

Alternative: Using the reverse triangle inequality, one obtains

$$|p(z)| = \left|\sum_{k=0}^{n} a_k \cdot z^k\right| \geq \left| |a_n| \cdot |z|^n - \left|\sum_{k=0}^{n-1} a_k \cdot z^k\right| \right|.$$

Setting $M = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$, one obtains for the second term (using the triangle inequality) for $|z| > 1$:

$$\left|\sum_{k=0}^{n-1} a_k \cdot z^k\right| \leq \sum_{k=0}^{n-1} |a_k| \cdot |z|^k \leq M \cdot \sum_{k=0}^{n-1} |z|^k \leq M \cdot n \cdot |z|^{n-1}.$$

Thus, whenever $|z| > \max\{1, M \cdot n/|a_n|\}$, we have

$$|p(z)| \geq |a_n| \left(|z| - \frac{M \cdot n}{|a_n|}\right) \cdot |z|^{n-1}$$

and $|p(z)| \to \infty$ as $|z| \to \infty$.

Remark: We can also uses the characterisation of singularities at the point of infinity ∞ (i.e., in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$): A function $f(z)$ has a removable singularity/pole/essential singularity at ∞ iff $f(1/z)$ has a removable singularity/pole/essential singularity at 0.

Please turn over!

Using this or the characterisation in Theorem V.1.1, it is easy to see that a non-constant polynomial has a pole at ∞. In fact, one can also show the converse: An entire function is a polynomial iff it has a pole (or, if it is constant, a removable singularity) at ∞. Entire functions such as the exponential function, sine, cosine are also called (entire) *transcendental function*. They have an essential singularity at ∞ (compare Theorem V.1.1), see the Question 1 on Exercise sheet 10 or the examples in the lecture.

2.) Find the Laurent series expansions of $f(z) = \frac{1}{z(1-z)(2-z)}$ for the annuli

$$\text{(i)}\quad 0 < |z| < 1, \qquad \text{(ii)}\quad 1 < |z| < 2, \qquad \text{(iii)}\quad 2 < |z|.$$

Hint: Do not compute integrals.

Solution: We write $f(z) = \frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2(1-\frac{z}{2})}\right)$ and use geometric series.

- $0 < |z| < 1$: In this case $|z| < 1$ and $\frac{|z|}{2} < 1$ so that

$$f(z) = \frac{1}{z}\left(\sum_{k=0}^{\infty} z^k - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k\right) = \sum_{k=-1}^{\infty}\left(1 - 2^{-(k+2)}\right) z^k.$$

- $1 < |z| < 2$: In this case $\frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1$ so that

$$\begin{aligned} f(z) &= -\frac{1}{z^2}\frac{1}{1-\frac{1}{z}} - \frac{1}{2z}\frac{1}{1-\frac{z}{2}} = -\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{1}{z^k} - \frac{1}{2z}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k \\ &= -\sum_{k=-\infty}^{-2} z^k - \sum_{k=-1}^{\infty} 2^{-(k+2)} z^k. \end{aligned}$$

- $2 < |z|$: Now $\frac{1}{|z|} < 1$ and $\frac{2}{|z|} < 1$ and we get

$$\begin{aligned} f(z) &= -\frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z}} - \frac{1}{1-\frac{2}{z}}\right) = \frac{1}{z^2}\left(\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - \sum_{k=0}^{\infty}\frac{1}{z^k}\right) \\ &= \sum_{k=-\infty}^{-2}\left(2^{-(k+2)} - 1\right) z^k. \end{aligned}$$

Note how the Laurent series expansions of the very same function are different for different annuli!

Note: you can also use the partial fraction expansion

$$f(z) = \frac{1}{2z} + \frac{1}{1-z} + \frac{1}{2(z-2)}.$$

Please turn over!

More precisely, you may use

$$f(z) = \frac{1}{2}\frac{1}{z} + \frac{1}{1-z} - \frac{1}{4}\frac{1}{(1-\frac{z}{2})}$$

$$= \frac{1}{2}\frac{1}{z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{4}\frac{1}{(1-\frac{z}{2})} = \frac{1}{2}\frac{1}{z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2z}\frac{1}{(1-\frac{2}{z})}$$

for parts (i), (ii), (iii), respectively.

3.) Find the Laurent series expansions of $f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$ for the annuli

(i) $0 < |z| < 1$, (ii) $0 < |z-1| < 1$, (iii) $0 < |z-2| < 1$.

Hint: Do not compute integrals.

Solution: We use geometric series throughout, rewriting the function suitably.

- $0 < |z| < 1$: Write $f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$; since $|z|, |\frac{z}{2}| < 1$ we get

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} z^k + \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = \frac{1}{z} + \sum_{k=0}^{\infty}\left(1 + \frac{1}{2^{k+1}}\right) z^k.$$

- $0 < |z-1| < 1$: Write $f(z) = \frac{1}{1-(1-z)} - \frac{1}{z-1} + \frac{1}{1-(z-1)}$; since $|z-1| = |1-z| < 1$ we get

$$f(z) = \sum_{k=0}^{\infty}(1-z)^k - \frac{1}{z-1} + \sum_{k=0}^{\infty}(z-1)^k = -\frac{1}{z-1} + \sum_{j=0}^{\infty} 2(z-1)^{2j}.$$

- $0 < |z-2| < 1$: Write $f(z) = \frac{1}{2}\frac{1}{1-\frac{(2-z)}{2}} - \frac{1}{1-(2-z)} - \frac{1}{z-2}$; since $|z-2|, |\frac{z-2}{2}| < 1$ we get

$$f(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(2-z)^k}{2^k} - \sum_{k=0}^{\infty}(2-z)^k - \frac{1}{z-2} = -\frac{1}{z-2} - \sum_{k=0}^{\infty}(-1)^k\left(1 - \frac{1}{2^{k+1}}\right)(z-2)^k.$$

What are the (nonzero) residues of f?

4.) Let $f : D \to \mathbb{C}$ be holomorphic. We say that $z_0 \in D$ is a *zero of order* $m \in \mathbb{N}$ of f if the Taylor series expansion of f at z_0

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k, \qquad \text{where} \qquad a_m \neq 0.$$

Prove that $z_0 \in D$ is a zero of order $m \in \mathbb{N}$ iff there is a holomorphic function g with $g(z_0) \neq 0$ so that $f(z) = (z - z_0)^m g(z)$.

Conclude that the zeros of a nonzero holomorphic function are isolated.

Solution: This is pretty much a book keeping exercise.

First assume that $z_0 \in D$ is a zero of order m of f, that is, its Taylor expansion starts with the m-th term:

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k.$$

The radius of convergence R of the Taylor series satisfies $R \geq \varrho > 0$, where $B_\varrho(z_0) \subset D$ (Cauchy-Taylor Theorem). Thus

$$g(z) = \frac{1}{(z - z_0)^m} \sum_{k=m}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k$$

defines, as a power series with radius of convergence $R > 0$, a holomorphic function on some $B_R(z_0)$ (by Theorem IV.2.2).

Conversely, if g is holomorphic in some $B_\varrho(z_0)$, $\varrho > 0$, with $g(z_0) \neq 0$ then

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k \qquad \text{on} \qquad B_\varrho(z_0), \qquad \text{where} \qquad b_0 \neq 0,$$

by the Cauchy-Taylor Theorem. Now, if $f(z) = (z - z_0)^m g(z)$, then the Taylor expansion of f at z_0

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = f(z) = (z - z_0)^m \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

so that $a_k = 0$ for $k = 0, \ldots, m-1$ and $a_m = b_0 \neq 0$ since the Taylor expansion is unique (see the Identity Theorem for power series).

Finally (to show that z_0 is isolated), if $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$, g holomorphic around z_0, then there is $R > 0$ so that $g(z) \neq 0$ for $z \in B_R(z_0)$ (why can we assume this?). Thus $f(z) \neq 0$ on $B_R(z_0) \setminus \{z_0\}$. (The last part can also be proven from the Identity Theorem for power series by contraposition: Suppose z_0 is not isolated, then show that the function f is the zero-function; so z_0 is not a zero of order m).

Optional question:

5.) Convince yourself that $z \mapsto \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$, i.e., it is holomorphic in $\mathbb{C}$ except for poles.

Solution: Clearly $\frac{1}{\sin z}$ is holomorphic on

$$D = \{z \in \mathbb{C} \,|\, \sin z \neq 0\} = \{n\pi \,|\, n \in \mathbb{Z}\},$$

and all singularities $n\pi$ of $\frac{1}{\sin z}$ are isolated. Since sin is periodic,

$$\sin(z + n\pi) = (-1)^n \sin z,$$

it suffices to investigate the function at $z = 0$. Since

$$\lim_{z\to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \dots} = 1$$

the function $B^*_\pi(0) \ni z \mapsto g(z) = \frac{z}{\sin z}$ extends holomorphically to $B_\pi(0)$ (e.g., by Theorem V.1.1(i) and hence

$$\frac{1}{\sin z} = \frac{1}{z} g(z) = \sum_{k=-1}^{\infty} a_{k+1} z^k$$

on $B^*_\pi(0)$, where $g(z) = \sum_{k=0}^{\infty} a_k z^k$ is the Taylor expansion of $g(z)$ on $B_\pi(0)$. Consequently, $\frac{1}{\sin z}$ has a simple pole at $z = 0$ and therefore, at $z = n\pi$, $n \in \mathbb{Z}$.