MA30056: Complex Analysis

Exercise Sheet 8: Power series

Please hand solutions in at the lecture on Monday 20th April.

Note: There will be no lecture, no drop-in session and no problem class next week (23 March – 27 March)!

Have a nice Easter break!

1.) Let $f_k : D \to \mathbb{C}$, $k \in \mathbb{N}$ and $D \subset \mathbb{C}$ *any* subset of $\mathbb{C}$, satisfy $|f_k(z)| \leq M_k$ for all $k \in \mathbb{N}$ and $z \in D$ and assume that $\sum_{k=0}^{\infty} M_k$ converges. Prove that $\sum_{k=0}^{\infty} f_k$ converges absolutely and uniformly on D (i.e., the Weierstrass M-test).
Hint: Use the Cauchy criterion.

2.) Let f be an entire function and z_0 arbitrary. Show that the Taylor series expansion of f at z_0 has radius of convergence $R = \infty$.

3.) Find the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ and determine its radius of convergence.
Hint: Do not compute derivatives.

4.) Show (without calculations): Given the complex sine and cosine, we have $\cos^2 z + \sin^2 z = 1$.
Also argue that the compound angle formulae (e.g., $\cos(z + w) = \cos z \cos w - \sin z \sin w$ with $z, w \in \mathbb{C}$) hold for the complex sine and cosine.

5.) Suppose that $x^2 + y^2 \leq 1$. Prove that $(x^2 - y^2 - 1)^2 + 4x^2y^2$ attains its maximum value when $x = 0$, $y = \pm 1$.

6.) Ask at least one question about something you have not understood in this unit.

Please turn over!

An interesting (although long) exercise:

7.) In this exercise we show that there is **no** holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.

(i) Consider the two Möbius transformations φ and ψ given by

$$\varphi(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z} \qquad \text{and} \qquad \psi(z) = \frac{z - \frac{3}{4}}{1 - \frac{3}{4}z}.$$

- Show: φ and ψ are holomorphic in $B_1(0)$ (and continuous on $\partial B_1(0)$).
- Show: $\varphi(\partial B_1(0)) = \partial B_1(0)$ and $\psi(\partial B_1(0)) = \partial B_1(0)$, i.e., φ and ψ map the unit circle on the unit circle.
  Hint: Note that three distinct (noncollinear) points determine a circle.
- Conclude that φ and ψ map $B_1(0)$ into itself.
  Hint: Maximum Modulus Theorem.

(ii) Prove the so-called *Schwarz' Lemma*[1]:
Suppose $\Phi : B_1(0) \to B_1(0)$ with $\Phi(0) = 0$ (i.e., Φ maps the unit disk into the unit disk and the origin to the origin), then $|\Phi(z)| \leq |z|$ for all $z \in B_1(0)$ and $|\Phi'(0)| \leq 1$.
Furthermore, if $|\Phi'(0)| = 1$ or $|\Phi(z)| = |z|$ for some $z \in B_1^*(0)$, then Φ is a rotation: $\Phi(z) = e^{i\theta} \cdot z$ for some real constant θ.

Hint: Apply the Maximum Modulus Theorem to the function

$$g(z) = \begin{cases} \Phi(z)/z & \text{if } z \in B_1^*(0) \\ \Phi'(0) & \text{if } z = 0. \end{cases}$$

Here, $B_1^*(0) = B_1(0) \setminus \{0\}$ denotes the punctured unit disk.

(iii) Now show that there is no holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.
Hint: Suppose such a function exists and consider $\Phi = \psi \circ f \circ \varphi$.

Optional questions:

8.) Let $f_n : B_1(0) \to \mathbb{C}$, $z \mapsto f_n(z) = \frac{z^{n+1}}{n+1}$. Prove that $f_n \to 0$ on $B_1(0)$ and $f_n' \to 0$ locally uniformly but not uniformly on $B_1(0)$.

9.) Let $B_1(0)$ be the open unit disc. Can you find a nonzero analytic function on $B_1(0)$ that has infinitely many zeros in $B_1(0)$? If yes, give your example and prove your claim; if not, give your reasoning why not.

[1] Not to be confused with the earlier statement by the same name that the second mixed partial derivatives commute.