MA30056: Complex Analysis

Exercise Sheet 8: Power series

Please hand solutions in at the lecture on Monday 20th April.

Note: There will be no lecture, no drop-in session and no problem class next week (23 March – 27 March)!

Have a nice Easter break!

1.) Let $f_k : D \to \mathbb{C}$, $k \in \mathbb{N}$ and $D \subset \mathbb{C}$ *any* subset of $\mathbb{C}$, satisfy $|f_k(z)| \leq M_k$ for all $k \in \mathbb{N}$ and $z \in D$ and assume that $\sum_{k=0}^{\infty} M_k$ converges. Prove that $\sum_{k=0}^{\infty} f_k$ converges absolutely and uniformly on D (i.e., the Weierstrass M-test).
Hint: Use the Cauchy criterion.

Solution: Fix $\varepsilon > 0$. Since $\sum_{k=0}^{\infty} M_k$ converges (absolutely since all $M_k \geq 0$) there is $N \in \mathbb{N}$ so that

$$\sum_{k=m}^{n} M_k < \varepsilon \qquad \text{whenever} \qquad n > m \geq N.$$

Then, for $n > m \geq N$ and all $z \in D$, we have

$$|\sum_{k=m}^{n} f_k(z)| \leq \sum_{k=m}^{n} |f_k(z)| \leq \sum_{k=m}^{n} M_k < \varepsilon,$$

so that, by the Cauchy criterium for uniform convergence (Theorem IV.1.1), $\sum_{k=0}^{\infty} f_k$ converges uniformly on D.

2.) Let f be an entire function and z_0 arbitrary. Show that the Taylor series expansion of f at z_0 has radius of convergence $R = \infty$.

Solution: Let D denote the domain of f, then f being entire means that its domain $D = \mathbb{C}$; thus any disk $B_R(z_0) \subset D$, $R > 0$ arbitrary. Hence, by the Cauchy-Taylor Theorem (Theorem IV.2.3), f equals its Taylor series (which, in particular, converges) on $B_R(z_0)$ for arbitrary $R > 0$, i.e., the radius of convergence is $R = \infty$.

3.) Find the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ and determine its radius of convergence.
Hint: Do not compute derivatives.

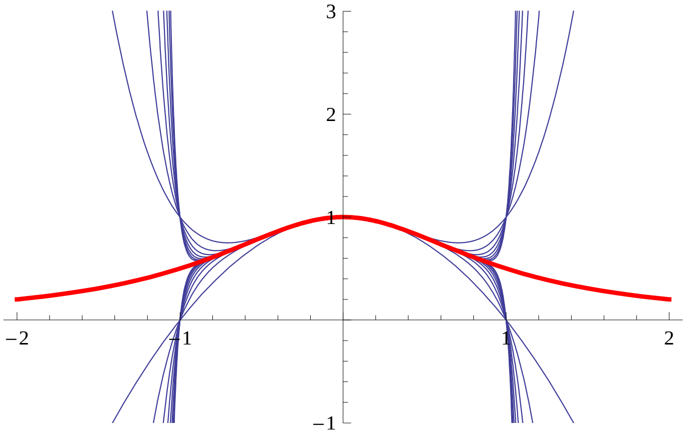
Solution: We write $f(z)$ as a geometric series:

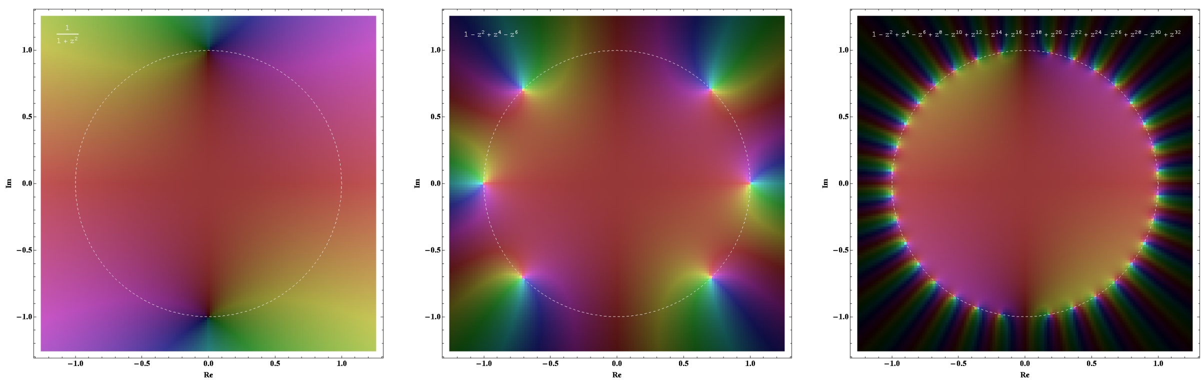
$$f(z) = \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty}(-z^2)^k = \sum_{k=0}^{\infty}(-1)^k z^{2k}.$$

$f(z)$ is defined on $\mathbb{C} \setminus \{\pm i\}$. Thus the radius of convergence of the power series with $z_0 = 0$ is $R = 1$ (by Cauchy-Taylor we have $R \geq 1$ and since f cannot be holomorphically extended through $\pm i$ we have $R \leq 1$).

We compare the original function f with the partial functions, first along the real line and then in $\mathbb{C}$.

Picture to the right: The graph of $f(x) = \frac{1}{1+x^2}$ is given in (thick) red for real values $x \in [-2, 2]$. The blue-ish (or violet) graphs are the partial sums $\sum_{k=0}^{N}(-1)^k x^{2k}$ for the same real numbers and for $1 \leq N \leq 15$. Clearly, there is no convergence of these partial sums outside the interval $[-1, 1]$.





The function $f(z) = \frac{1}{1+z^2}$ on the left, and two partial sums of its Taylor series around 0, namely up to order 6 in the middle and up to order 32 on the right. Observe that the Taylor series converges in the unit disk to f, but not outside.

For a "movie" on the convergence of the Taylor series in question, visit (the three pics above are taken from this movie)

http://www.maths.bath.ac.uk/~bs259/ma30056/movie.html

4.) Show (without calculations): Given the complex sine and cosine, we have $\cos^2 z + \sin^2 z = 1$.
Also argue that the compound angle formulae (e.g., $\cos(z+w) = \cos z \cos w - \sin z \sin w$ with $z, w \in \mathbb{C}$) hold for the complex sine and cosine.

Solution: We note that both the complex sine and the complex cosine are entire. Set $f(z) = \big(\cos^2 z + \sin^2 z\big) - 1$. Then, f is holomorphic on $\mathbb{C}$ (i.e., entire) and satisfies $f(z) = 0$ for all real z, i.e., all $\{z \in \mathbb{C} \,|\, \operatorname{Im} z = 0\}$. By the Identity Theorem for holomorphic functions (Theorem IV.3.2), one has $f \equiv 0$ (e.g., take any real sequence $(z_n)_n$ converging to 0), which establishes the claim.

The compound angle formulae can be established by a two-stage process of the same kind: Firstly, fix w to be a real number and define $f_w(z) = \cos(z+w) - (\cos z \cos w - \sin z \sin w)$. Then, using the argument as above, $f_w \equiv 0$ (since $f_w(z) = 0$ for all real z), i.e., $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all $z \in \mathbb{C}$ and all $w \in \mathbb{R}$. Secondly, fix $z \in \mathbb{C}$ and define $g_z(w) = \cos(z+w) - (\cos z \cos w - \sin z \sin w)$. Here, we now know that $g_z(w) = 0$ for all real w (and all $z \in \mathbb{C}$), and it also follows $g_z \equiv 0$, wherefore $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all $z, w \in \mathbb{C}$.

5.) Suppose that $x^2 + y^2 \leq 1$. Prove that $(x^2 - y^2 - 1)^2 + 4x^2y^2$ attains its maximum value when $x = 0$, $y = \pm 1$.

Solution: Note that the modulus function of $f(x + i\,y) = f(z) = z^2 - 1 = x^2 + 2i\,xy - y^2 - 1$ is $|f|(x + i\,y) = \sqrt{(x^2 - y^2 - 1)^2 + (2xy)^2}$, so that $(x^2 - y^2 - 1)^2 + 4x^2y^2$ is the square of the modulus of f (clearly, $|f|$ has a maximum iff $|f|^2$ has a maximum). We also note that $|f| : \mathbb{C} \to \mathbb{R}$ is continuous as composition of the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ and the continuity of the modulus $|\cdot| : \mathbb{C} \to \mathbb{R}$.

Let D be a bounded domain, e.g., $D = B_1(0)$. Then its closure $\overline{D}$ (e.g., $\overline{B_1}(0)$) is compact, and we note (by abuse of notation we call all functions here $|f|$ even though they are defined on different sets) the following properties:

(i) The function $|f| : \overline{D} \to \mathbb{R}$ has compact image $|f|(\overline{D})$ (compare Exercise sheet 2, Question 4) and therefore attains its maximum for a $z_0 \in \overline{D}$.

(ii) If the function $|f| : D \to \mathbb{R}$ has a maximum at $z_0 \in D$, then f is constant by the Maximum Value Theorem.

Obviously, the function $f(z) = z^2 - 1$ is not constant, therefore it has no maximum in $B_1(0)$. The function f must therefore achieve its maximum in $\overline{B_1}(0)$ on the boundary $\partial B_1(0)$, i.e., for a z with $|z| = 1$.

Now, either chose a (one-dimensional) parametrisation (e.g., $\gamma(t) = \cos t + i\,\sin t$ where $t \in [0, 2\pi]$) of the unit circle and calculate the maximum $|f|(\gamma(t))$ with the usual one-dimensional techniques (derivatives etc.), or observe that the unit circle is also given by $\{x + i\,y \mid x^2 = 1 - y^2,\ y \in [-1, 1]\}$. Then, on the unit circle we have

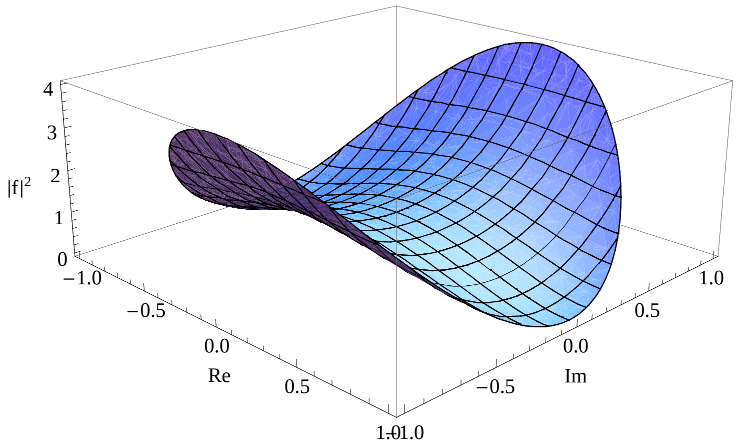
$$(x^2 - y^2 - 1)^2 + 4x^2y^2 \overset{x^2=1-y^2}{=} 4y^4 + 4y^2 - 4y^4 = 4y^2,$$

and the maximum 4 for $y \in [-1, 1]$ is attained for $y = \pm 1$ and $x = 0$. This establishes the claim.

Note: Realizing that a holomorphic function is hidden here, saves us from applying the techniques of multivariable analysis (so we do not have to calculate any partial derivatives here)!

Note: That the maximum modulus of a (nonconstant) holomorphic function is attained on the boundary of the domain is the surprising property here (compare this with real analysis).

Picture to the right: The graph of $|f|^2(x + i\,y) = (x^2 - y^2 - 1)^2 + 4x^2y^2$ on $\overline{B_1(0)}$ has the shape of a saddle.



6.) Ask at least one question about something you have not understood in this unit.

An interesting (although long) exercise:

7.) In this exercise we show that there is **no** holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.

(i) Consider the two Möbius transformations φ and ψ given by

$$\varphi(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z} \qquad \text{and} \qquad \psi(z) = \frac{z - \frac{3}{4}}{1 - \frac{3}{4}z}.$$

- Show: φ and ψ are holomorphic in $B_1(0)$ (and continuous on $\partial B_1(0)$).
- Show: $\varphi(\partial B_1(0)) = \partial B_1(0)$ and $\psi(\partial B_1(0)) = \partial B_1(0)$, i.e., φ and ψ map the unit circle on the unit circle.
  Hint: Note that three distinct (noncollinear) points determine a circle.

Please turn over!

- Conclude that φ and ψ map $B_1(0)$ into itself.
  Hint: Maximum Modulus Theorem.

(ii) Prove the so-called *Schwarz' Lemma*[1]:
Suppose $\Phi : B_1(0) \to B_1(0)$ is holomorphic with $\Phi(0) = 0$ (i.e., Φ maps the unit disk into the unit disk and the origin to the origin), then $|\Phi(z)| \leq |z|$ for all $z \in B_1(0)$ and $|\Phi'(0)| \leq 1$.
Furthermore, if $|\Phi'(0)| = 1$ or $|\Phi(z)| = |z|$ for some $z \in B_1^*(0)$, then Φ is a rotation: $\Phi(z) = \mathrm{e}^{i\theta} \cdot z$ for some real constant θ.

Hint: Apply the Maximum Modulus Theorem to the function

$$g(z) = \begin{cases} \Phi(z)/z & \text{if } z \in B_1^*(0) \\ \Phi'(0) & \text{if } z = 0. \end{cases}$$

Here, $B_1^*(0) = B_1(0) \setminus \{0\}$ denotes the punctured unit disk.

(iii) Now show that there is no holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.
Hint: Suppose such a function exists and consider $\Phi = \psi \circ f \circ \varphi$.

Solution:

(i) The denominator of φ vanishes at -2, while the one of ψ vanishes at $\frac{4}{3}$. Thus, by the algebra of holomorphic functions (see Theorem II.2.2), φ is holomorphic on $B_2(0)$ while ψ is holomorphic on $B_{\frac{4}{3}}(0)$; in particular, they are holomorphic in $B_1(0)$ and continuous on $\partial B_1(0)$.
As Möbius transformations (see Question 2 on Exercise sheet 2), φ and ψ map circles to cirles or lines. Noting that $\varphi(1) = 1$, $\psi(1) = 1$, $\varphi(-1) = -1$, $\psi(-1) = -1$, $\varphi(i) = \frac{1}{5}(4 - 3i)$, $\psi(i) = \frac{1}{25}(-24 + 7i)$ lie on the unit circle, we obtain that both φ and ψ map the unit circle to the unit circle.
By the Maximum Modulus Theorem (compare previous exercise), the absolute value of these (nonconstant) functions φ, ψ on $B_1(0)$ is attained on the boundary $\partial B_1(0)$, i.e., the unit circle. But since φ, ψ map the unit circle into the unit circle, this maximum modulus is 1. Consequently, $\varphi, \psi : B_1(0) \to B_1(0)$.

(ii) Note that (strictly speaking) Φ is not defined on the unit circle $\partial B_1(0)$, thus we apply the Maximum Modulus Theorem on smaller disks[2] as follows:
First note that g is holomorphic on $B_1(0)$: Φ is holomorphic in $B_1(0)$ and thus has a Taylor series expansion $\sum_{k=0}^{\infty} a_k z^k$ there; moreover, since $\Phi(0) = 0$,

Please turn over!

[1] Not to be confused with the earlier statement by the same name that the second mixed partial derivatives commute.

[2] Also note that by Cauchy's inequalities (Corollary III.3.5) we have $|\Phi'(0)| \leq 1$ (noting that $|f(z)| \leq 1$ on $B_1(0)$, take $\Gamma = \partial B_r(0) \subset B_1(0)$ and let $r \to 1$)).

its constant term $a_0 = 0$. Thus, g has Taylor series expansion $\sum_{k=0}^{\infty} a_k z^{k-1} = \sum_{\ell=0}^{\infty} a_{\ell+1} z^\ell$ on $B_1(0)$ and is therefore holomorphic there (check that $g(0) = \Phi'(0)$).

Now, suppose $r \in (0,1)$. It is clear that $|g(z)| = \left|\frac{\Phi(z)}{z}\right| \leq \left|\frac{1}{r}\right|$ for $z \in \partial B_r(0)$, and so the Maximum Modulus Theorem shows (compare again the previous exercise and note that $1 \leq \frac{1}{r}$) that $|g(z)| \leq \frac{1}{r}$ in $B_r(0)$. Since this holds for all $r \in (0,1)$, it follows that $|g(z)| \leq 1$ on $B_1(0)$ and hence that $|\Phi(z) \leq |z|$.

Finally, if we have $|\Phi'(0)| = 1$ or $|\Phi(z)| = |z|$ for some nonzero $z \in B_1(0)$, then we have $|g(z)| = 1$ for some $z \in B_1(0)$ and hence the Maximum Modulus Theorem shows that g is constant.

(iii) We show this by contraposition, i.e., we suppose that $f : B_1(0) \to B_1(0)$ is holomorphic with $f(\frac{1}{2}) = \frac{3}{4}$ and show that this implies that $f'(\frac{1}{2}) \neq \frac{2}{3}$.

To this end, set $\Phi(z) = (\psi \circ f \circ \varphi)(z)$ for $z \in B_1(0)$. Then, $\Phi : B_1(0) \to B_1(0)$ is holomorphic and we have $\Phi(0) = (\psi \circ f \circ \varphi)(0) = (\psi \circ f)(\frac{1}{2}) = \psi(\frac{3}{4}) = 0$. Thus, by Schwarz' Lemma, we get that $|\Phi'(0)| \leq 1$.

By the chain rule, we have

$$\begin{aligned}\Phi'(0) = (\psi \circ f \circ \varphi)'(0) &= (\psi' \circ f \circ \varphi)(0) \cdot (f' \circ \varphi)(0) \cdot \varphi'(0) \\ &= \psi'(\tfrac{3}{4}) \cdot f'(\tfrac{1}{2}) \cdot \varphi'(0) = \frac{16}{7} \cdot f'(\tfrac{1}{2}) \cdot \frac{3}{4}.\end{aligned}$$

Thus, we have $|\Phi'(0)| = \frac{12}{7} \cdot |f'(\frac{1}{2})|$, and therefore obtain the estimate $|f'(\frac{1}{2})| = \frac{7}{12}\,|\Phi'(0)| \leq \frac{7}{12} \approx 0.5833 < \frac{3}{5}$.

So, if $f : B_1(0) \to B_1(0)$ is a holomorphic function with $f(\frac{1}{2}) = \frac{3}{4}$, then $f'(\frac{1}{2}) \leq \frac{7}{12}$ while $f'(\frac{1}{2}) = \frac{2}{3}$ is not possible.

Optional questions:

8.) Let $f_n : B_1(0) \to \mathbb{C}$, $z \mapsto f_n(z) = \frac{z^{n+1}}{n+1}$. Prove that $f_n \to 0$ on $B_1(0)$ and $f_n' \to 0$ locally uniformly but not uniformly on $B_1(0)$.

Solution: If $z \in B_1(0) = \{z \in \mathbb{C} \,|\, |z| < 1\}$ then

$$|f_n(z)| = \left|\frac{z^{n+1}}{n+1}\right| < \frac{1}{n+1}.$$

Thus, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $\frac{1}{N} \leq \varepsilon$ to find that

$$|f_n(z) - 0| < \frac{1}{n+1} < \frac{1}{N} \leq \varepsilon \qquad \text{whenever} \qquad n \geq N,$$

that is, $f_n \to 0$ in $B_1(0)$.

Now $f_n' \to 0$ locally uniformly by Theorem IV.1.4.

To prove this by bare hands, fix $z_0 \in B_1(0)$ and $\varrho = \frac{1}{2}(1 - |z_0|) > 0$. Write $r = |z_0| + \varrho$. Then, whenever $z \in \overline{B}_\varrho(z_0) \subset \overline{B}_r(0)$,

$$|f_n'(z) - 0| = |z^n| \leq r^n.$$

Fix $\varepsilon > 0$. Since $r^n \to 0$ as $n \to \infty$ there is $N \in \mathbb{N}$ so that

$$|f_n'(z)| \leq r^n < \varepsilon \qquad \text{whenever} \qquad n \geq N,$$

that is, $f_n' \to 0 = f'$ on $\overline{B}_\varrho(z_0)$ (in fact, on $\overline{B}_r(0)$).

To see that $f_n' \to 0$ not uniformly on $B_1(0)$, take $z_n = 1 - \frac{1}{2n}$ and observe that

$$|f_n'(z_n)| = \left(1 - \frac{1}{2n}\right)^n \geq 1 - n\,\frac{1}{2n} = \frac{1}{2}$$

by Bernoulli's inequality. Hence,

$$\exists \varepsilon = \frac{1}{2}\, \forall N \in \mathbb{N}\, \exists n = N\, \exists z = z_n : |f_n'(z) - 0| \geq \varepsilon,$$

that is, $f_n' \to 0$ not uniformly in $B_1(0)$.

Another possibility is to take $z_n = 1 - \frac{1}{n}$ and observe that

$$\left(1 - \frac{1}{n}\right)^n \to \mathrm{e}^{-1} \neq 0$$

as, in fact, we know that $(1 + \frac{x}{n})^n \to \mathrm{e}^x$.

9.) Let $B_1(0)$ be the open unit disc. Can you find a nonzero analytic function on $B_1(0)$ that has infinitely many zeros in $B_1(0)$? If yes, give your example and prove your claim; if not, give your reasoning why not.

Solution: Let $f(z) = \sin\left(\frac{z+1}{z-1}\right)$. Then, $f = \sin \circ g$ where $g(z) = \frac{z+1}{z-1}$ and thus is holomorphic in $\mathbb{C} \setminus \{1\}$. In particular, $f(z)$ is holomorphic in $B_1(0)$. Also, it is clear that $f(z)$ is not identically zero.

But $f(z) = 0 \Rightarrow \sin\left(\frac{z+1}{z-1}\right) = 0 \Rightarrow \frac{z+1}{z-1} = -n\,\pi$ for $n \in \mathbb{Z}$ (we include an "unnecessary" minus sign here to make the next calculation clearer). Thus, f vanishes at $z_n = \frac{n\,\pi - 1}{n\,\pi + 1}$ where $n \in \mathbb{Z}$. For positive n, $z_n \in B_1(0)$ (in fact, they are real numbers with $0 < z_n < 1$ for $n \geq 1$) and they are all distinct. So, the nonzero function $f(z) = \sin\left(\frac{z+1}{z-1}\right)$ has infinitely many zeros in $B_1(0)$.

With regard to the Identity Theorem for holomorphic functions (Theorem IV.3.2) note that the limit $\lim_{n\to\infty} z_n = 1$ does not belong to the domain $B_1(0)$ on which f is holomorphic!