MA30056: Complex Analysis

Exercise Sheet 7: Applications and Sequences of Complex Functions

Please hand solutions in at the lecture on Monday 16th March.

1.) Prove Gauss' Fundamental Theorem of Algebra.
Hint: Suppose that a polynomial $p(z)$ has no zeroes and conclude that $f(z) = \frac{1}{p(z)}$ is bounded.)

Solution: We proof this by contraposition, i.e., since we want to proof that a nonconstant polynomial has at least one zero, we actually show that a polynomial without a zero is constant.

Suppose the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, has no zeroes. Then $f(z) = \frac{1}{p(z)}$ defines an entire function.

We wish to show that f is bounded:

(i) Since $\frac{p(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n} \to 1$ as $|z| \to \infty$, there is $R > 0$ so that

$$|z| > R \qquad \Rightarrow \qquad \left|\frac{p(z)}{a_n z^n}\right| \geq \frac{1}{2}.$$

Therefore, we have $R > 0$ so that

$$|z| > R \qquad \Rightarrow \qquad |f(z)| \leq \frac{2}{|a_n z^n|} \leq \frac{2}{|a_n|\, R^n}.$$

(ii) Since $\overline{B}_R(0)$ is compact and f is continuous, there is $M \in \mathbb{R}$ so that

$$|z| \leq R \qquad \Rightarrow \qquad |f(z)| \leq M.$$

Thus f is bounded (by $\max\{\frac{2}{|a_n|\, R^n}, M\}$). Hence, by Liouville's Theorem, f is constant – and so is p.

2.) Evaluate $\int_{|z|=1} \frac{e^{i\alpha z}}{(z-z_0)^2}\, dz$ for $\alpha > 0$ and

$$\text{(i)} \quad |z_0| < 1, \qquad \qquad \text{(ii)} \quad |z_0| > 1.$$

Solution:

(ii) If $|z_0| > 1$ then $z \mapsto \frac{e^{i\alpha z}}{(z-z_0)^2}$ is holomorphic in the interior $B_1(0)$ of the circle $\{z \,|\, |z| = 1\}$. Hence, by Cauchy's Theorem,

$$\int_{|z|=1} \frac{e^{i\alpha z}}{(z-z_0)^2}\, dz = 0.$$

Please turn over!

(i) If $|z_0| < 1$ then we apply Cauchy's formula for the first derivative to obtain

$$\int_{|z|=1} \frac{e^{i\alpha z}}{(z-z_0)^2}\, dz = \frac{2\pi i}{1!} \frac{d}{dz} e^{i\alpha z}\bigg|_{z=z_0} = 2i^2\alpha\pi\, e^{i\alpha z_0} = -2\alpha\pi\, e^{i\alpha z_0}\,.$$

3.) Noting that for the multifunction log on the cut plane $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$we have $\log z = \log|z| + i\,\mathrm{Arg}\,z + 2k\,\pi i$ and $\log(-z) = \log|z| + i\,\mathrm{Arg}\,z + (2k+1)\,\pi i$ where $k \in \mathbb{Z}$, find the mistake in the following argument:

$$\begin{aligned} & \log\left((-z)^2\right) = \log\left(z^2\right) \\ \Rightarrow\quad & \log(-z) + \log(-z) = \log z + \log z \\ \Rightarrow\quad & 2\log(-z) = 2\log z \\ \Rightarrow\quad & \log(-z) = \log z. \end{aligned}$$

Solution: First, we note that $\log z = \log|z| + i\,\mathrm{Arg}\,z + 2k\,\pi i$ for $k \in \mathbb{Z}$ is the sloppy way to actually say $\log z = \{\log|z| + i\,\mathrm{Arg}\,z + 2k\,\pi i \mid k \in \mathbb{Z}\}$ (so, instead of a single value as a function, a multifunction associates a whole set of values to a point).

We look at a little example: let $z = e^{i\pi/10}$. Then we have

$$-z = -e^{i\pi/10} = e^{i\pi\, 11/10} = e^{-i\pi\, 9/10}, \qquad z^2 = e^{i\pi/5} \quad \text{and} \quad (-z)^2 = e^{i\pi\, 11/5} = e^{i\pi/5}\,.$$

So, we have the following logarithms (note that all these numbers have modulus 1):

$$\log z = \left\{ i\pi\,\frac{1}{10} + 2\pi i\, k \mid k \in \mathbb{Z} \right\}, \qquad \log(-z) = \left\{ -i\pi\,\frac{9}{10} + 2\pi i\, k \mid k \in \mathbb{Z} \right\},$$

$$\text{and} \quad \log z^2 = \left\{ i\pi\,\frac{1}{5} + 2\pi i\, k \mid k \in \mathbb{Z} \right\} = \log(-z)^2.$$

So, certainly the first line in the above argument holds. For the second line, we have to make sense of $\log(-z)+\log(-z)$ (and also $\log z+\log z$) in this multifunction case: The values of a sum are obtained by adding each value of one set to each value of the other set, i.e.,

$$\begin{aligned} \log(-z) + \log(-z) &= \left\{ -i\pi\,\frac{9}{10} + 2\pi i\, k \mid k \in \mathbb{Z} \right\} + \left\{ -i\pi\,\frac{9}{10} + 2\pi i\, \ell \mid \ell \in \mathbb{Z} \right\} \\ &= \left\{ -i\pi\,\frac{9+9}{10} + 2\pi i\,(k+\ell) \mid k,\ell \in \mathbb{Z} \right\} \\ &\overset{(\star)}{=} \left\{ -i\pi\,\frac{9}{5} + 2\pi i\, m \mid m \in \mathbb{Z} \right\} \\ &\overset{m=\tilde{m}+1}{=} \left\{ i\pi\,\frac{1}{5} + 2\pi i\, \tilde{m} \mid \tilde{m} \in \mathbb{Z} \right\} \qquad (1) \\ &= \log(-z)^2, \end{aligned}$$

Please turn over!

where in $(\star)$ we make use of the fact that the sum of two integers is again an integer (and that every integer can be written as sum of two integers, i.e., that $\mathbb{Z}+\mathbb{Z}=\mathbb{Z}$). A similar calculation shows that $\log z + \log z = \log z^2$, so everything is fine by going from the first to the second line.

Now, look at the third line:

$$\begin{aligned} 2\,\log(-z) &= 2\left\{-i\pi\,\frac{9}{10} + 2\pi i\,k \mid k \in \mathbb{Z}\right\} \\ &= \left\{-i\pi\,\frac{9}{5} + 4\pi i\,k \mid k \in \mathbb{Z}\right\} \\ &= \left\{i\pi\,\frac{1}{5} + 2\pi i\,(2\,n-1) \mid n \in \mathbb{Z}\right\}, \end{aligned}$$

and comparing this set with the one in (1) above, we see that only the values with odd $\tilde{m}$ are in this set. Similarly, we have

$$\begin{aligned} 2\,\log z &= 2\left\{i\pi\,\frac{1}{10} + 2\pi i\,k \mid k \in \mathbb{Z}\right\} \\ &= \left\{i\pi\,\frac{1}{5} + 4\pi i\,k \mid k \in \mathbb{Z}\right\} \\ &= \left\{i\pi\,\frac{1}{5} + 2\pi i\,(2\,\tilde{n}) \mid \tilde{n} \in \mathbb{Z}\right\}, \end{aligned}$$

and here only the values of (1) with even $\tilde{m}$ are present. So, the third line is false (the sets on the left and right are not equal) and the mistake occurs in going from the second to the third line.

This holds generally: The mistake in the above "argument" occurs in going from the second to the third line! The sum $\log(-z) + \log(-z)$ cannot be replaced by $2\log(-z)$, since the sum in question is obtained from the set of values $\log(-z)$ (multifunction!) by adding each of these values to itself and to all the other values of the set $\log(-z)$, whereas the set $2\log(-z)$ is obtained by simply doubling all the numbers $\log(-z)$, i.e., by adding each such number to itself only. Therefore $\log(-z) + \log(-z) \neq 2\log(-z)$, and similarly, $\log z + \log z \neq 2\log z$.

Lesson learnt: *Be careful with multifunctions;-)*

Remark: The deal with multifunctions is that we look at all possible values (that, e.g., an "inverse" function like the logarithm or some root might have) *at once*! The following applet by Terence Tao nicely visualises what that is about (the principal branch is coloured red on the right – watch out how the colour "jumps" if you go in circles around the origin)

http://www.math.ucla.edu/~tao/java/Multi.html

By the way, Tao has won the Field's medal in 2006 (the highest honor in mathematics), mainly for proving the existence of arbitrarily long arithmetic progressions of prime numbers. This fact is a "converse" to the Classical Dirichlet Theorem which claims that there are infinite numbers of primes in certain arithmetic progressions.

4.) Calculate all possible values of i^i. What is the principal value of i^i?
Do the same for $i^{7/10}$.
State (without proof) under which conditions the power b^a is single-valued, has finitely many values or has infinitely many values.

Solution:

$$i^i = \mathrm{e}^{i \log i} = \mathrm{e}^{i\,(\log|i| + i\arg(i) + 2\pi i k)} = \mathrm{e}^{i\,(0 + i\frac{\pi}{2} + 2\pi i\,k)} = \mathrm{e}^{-\frac{\pi}{2} - 2\pi\,k},$$

where $k \in \mathbb{Z}$ (note that these are infinitely many real values). The principal value is $i^i = \mathrm{e}^{i\,\mathrm{Log}(i)} = \mathrm{e}^{-\frac{\pi}{2}}$.

$$i^{7/10} = \mathrm{e}^{7\cdot\log i/10} = \mathrm{e}^{7\,(\log|i| + i\arg(i) + 2\pi i k)/10} = \mathrm{e}^{7\,(0 + i\frac{\pi}{2} + 2\pi i\,k)/10} = \mathrm{e}^{\frac{7i\pi}{20}} \cdot \mathrm{e}^{7\,\frac{2\pi i\,k}{10}}$$

where $k \in \mathbb{Z}$. However, note that the set $\left\{\mathrm{e}^{7\,\frac{2\pi i\,k}{10}}\,|k \in \mathbb{Z}\right\}$ is exactly the set of all 10th roots of unity (and thus of cardinality 10), compare with Question 2 on Exercise sheet 1. The principal value is $\mathrm{e}^{\frac{7i\pi}{20}} \approx 0.4540 + 0.8910\,i$.

Looking closer at these two examples (and the examples given in the lecture and/or on Self-assessment sheet 7) suggest the following characterisation: For $b \neq 0$, the power b^a is single-valued iff $a \in \mathbb{Z}$, has finitely many values iff $a \in \mathbb{Q}$ (if $a = p/q$ is given in lowest terms, two such values are related via a factor of a qth root of unity), and has infinitely many values iff $a \in \mathbb{C} \setminus \mathbb{Q}$. For a proof of this claim, see [I. Stewart & D. Tall: Complex Analysis; Section 14.6].

5.) Prove that the limit function of a locally uniformly convergent sequence of continuous functions is continuous (i.e., Theorem IV.1.2).

Solution: Let $f_n : D \to \mathbb{C}$ be continuous and suppose $f_n \to f$ locally uniformly on D. We want to show that $f : D \to \mathbb{C}$ is continuous. This is done by the usual M11 argument with a slight twist.

Fix $\varepsilon > 0$.

Fix $z_0 \in D$ and $\varrho > 0$ so that $f_n \to f$ on $\overline{B}_\varrho(z_0) \subset D$ (by local uniform convergence). Then there is $N \in \mathbb{N}$ so that

$$\forall z \in \overline{B}_\varrho(z_0) : |f_n(z) - f(z)| < \frac{\varepsilon}{3}$$

whenever $n \geq N$.

Since f_N is continuous on $\overline{B}_\varrho(z_0) \subset D$ there is $\delta > 0$ so that

$$|z - z_0| < \delta \qquad \Rightarrow \qquad |f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$$

for any $z \in \overline{B}_\varrho(z_0)$.

Putting these together we obtain

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| < 3\frac{\varepsilon}{3} = \varepsilon$$

whenever $|z - z_0| < \min\{\delta, \varrho\}$, showing that f is continuous at $z_0 \in D$.

Optional questions:

6.) Determine the inverse function arccos of the complex cosine $\cos z = \frac{1}{2}\left(e^{i\,z} + e^{-i\,z}\right)$. Hence calculate all values arccos(3/2). Which one of these is the principle value?

Solution: We want to a function $f(z)$ such that $\cos f(z) = z$ for all $z \in \mathbb{C}$, i.e., $z = \frac{1}{2}\left(e^{i\,f(z)} + e^{-i\,f(z)}\right)$. Multiplying the last equation by $e^{i\,f(z)}$, we have to solve the equation

$$\left(e^{if(z)}\right)^2 - 2z\,e^{i\,f(z)} + 1 = 0$$

for $f(z)$. This yields

$$e^{i\,f(z)} = z + \left(z^2 - 1\right)^{\frac{1}{2}}$$

(note that the square root is a multifunction with (in general) two values, otherwise you might prefer to write this as $e^{i\,f(z)} = z \pm (z^2-1)^{\frac{1}{2}}$) and therefore

$$f(z) = -i\,\log\left(z + \left(z^2 - 1\right)^{\frac{1}{2}}\right) \tag{2}$$

(where now also log is a multifunction).

We check that $f(\cos z) = z = \cos f(z)$ (for all $z \in \mathbb{C}$) wherefore this is indeed the complex arccos.

We use Eq. (2) to calculate arccos(3/2):

$$\begin{aligned}\arccos(3/2) &= -i\,\log\left(\frac{3}{2} + \left(\left(\frac{3}{2}\right)^2 - 1\right)^{\frac{1}{2}}\right)\\ &= -i\,\log\left(\frac{3}{2} + \left(\frac{5}{4}\right)^{\frac{1}{2}}\right) \;=\; -i\,\log\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)\\ &\overset{(\star)}{=} -i\left(\log\left(\frac{3 \pm \sqrt{5}}{2}\right) + 2\pi i k\right) \;=\; -i\,\log\left(\frac{3 \pm \sqrt{5}}{2}\right) + 2\pi k\end{aligned}$$

with $k \in \mathbb{Z}$ (after the step "($\star$)" the function "log" denotes the usual real logarithm, before that step the multi-valued complex logarithm). The principal value is $-i\,\log\left(\frac{3+\sqrt{5}}{2}\right) \approx 0.9624\,i$.

Remark: We continue our remark on multifunctions after Question 3 here.
Instead of introducing multifunctions, one can define the complex logarithm (and then also the roots, the above mentioned arccos, etc.) over an appropriate *Riemann surface* M; that is, instead of defining the logarithm $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ as

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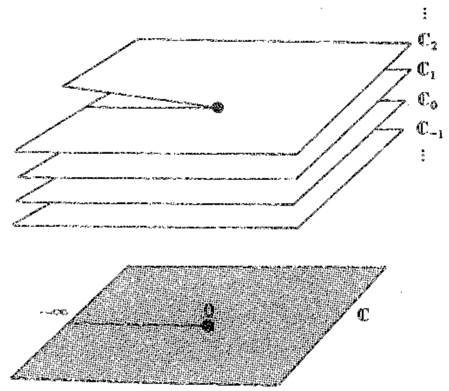
multifunction, the logarithm is defined by $\log : M \to \mathbb{C}$ over an appropriate Riemann surface M. For the complex logarithm, countably but infinitely many copies of $\mathbb{C}$ (named $\mathbb{C}_k$ with $k \in \mathbb{Z}$ in the upper part of the figure to the right) of the cut plane are pasted together "at the respective cut" to yield this Riemann surface M and allow the logarithm to vary continuously, see picture to the right.
This figure is taken from [I. Stewart & D. Tall: Complex Analysis; Fig. 14.8].



7.) Let $g : D \to A$ be continuous and $f_n \to f$ locally uniformly on A. Prove that $f_n \circ g \to f \circ g$ is locally uniformly on D (i.e., Lemma IV.1.3).

Solution: Let $z \in D$ and denote $w = g(z)$; fix $r > 0$ so that $f_n \to f$ on $B_r(w)$. Since $g : D \to A$ is continuous

$$\exists \varrho > 0 \, \forall w \in \overline{B}_\varrho(z) : g(w) \in \overline{B}_r(w).$$

Now take $\varepsilon > 0$.

Since $f_n \to f$ locally uniformly on A

$$\exists N \in \mathbb{N} \, \forall n \geq N \, \forall \omega \in \overline{B}_r(w) : |f_n(\omega) - f(\omega)| < \varepsilon.$$

Therefore we have $N \in \mathbb{N}$ so that

$$\forall n \geq N \, \forall w \in \overline{B}_\varrho(z) : |f_n(g(w)) - f(g(w))| < \varepsilon,$$

that is, we have $\varrho > 0$ so that $f_n \circ g \to f \circ g$ on $\overline{B}_\varrho(z)$.

8.) Show: $f_n(z) = \frac{1}{z-n}$ converges pointwise but not uniformly on $\mathbb{C} \setminus \mathbb{N}$ to the zero function. However, the sequence f_n converges uniformly on $B_1(0)$ to the zero function.

Solution: The sequence of functions $(f_n)_n$ converges pointwise to the zero function on $\mathbb{C} \setminus \mathbb{N}$ since, for any $\varepsilon > 0$, we have

$$\left| \frac{1}{z-n} \right| < \frac{1}{n - |z|}, \qquad \text{if } n > |z|,$$
$$< \varepsilon, \qquad \text{if } n > \frac{1}{\varepsilon} + |z|.$$

Please turn over!

It follows that $\lim_{n\to\infty} \frac{1}{z-n} = 0$ if $z \in \mathbb{C} \setminus \mathbb{N}$.

In other words, we have shown that it is possible to find a number N (dependent on z) such that, for any $\varepsilon > 0$, we have $1/(z-n) < \varepsilon$ if $n > N$. However, for a given $\varepsilon > 0$, there is no $N \in \mathbb{N}$ such that $\left|\frac{1}{z-n}\right| \leq \varepsilon$ if $n > N$ for *all* $z \in \mathbb{C} \setminus \mathbb{N}$: For a given $m \in \mathbb{N}$, the set $\mathbb{C} \setminus \mathbb{N}$ contains points z for which $|z-m| < \frac{1}{\varepsilon}$ (e.g., $z = m + \frac{1}{2\varepsilon}$) and thus $\frac{1}{|z-m|} > \varepsilon$.

The sequence of functions $(f_n)_n$ converges uniformly to the zero function on the open unit disk $B_1(0)$, since, given any $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{|z-n|} &< \frac{1}{n-1}, \qquad \text{if } n > 1, \\ &< \varepsilon, \qquad \text{if } n > \frac{1}{\varepsilon} + 1. \end{aligned}$$

We can therefore choose N to be the smallest natural number greater than $(1/\varepsilon)+1$. Notice that N depends on ε, but not on z.