MA30056: Complex Analysis

Exercise Sheet 6: Homotopy version of Cauchy's Theorem and Cauchy Formulae

Please hand solutions in at the lecture on Monday 9th March.

Note: There will be no drop-in session on Tuesday 3rd March.

1.) Prove that the function $f(z) = \frac{1}{z}$ has an anti-derivative F in the *cut plane* $\mathbb{C}\backslash\mathbb{R}_{\leq 0} = \{z = r\,\mathrm{e}^{i\varphi}\,|r > 0, -\pi < \varphi < \pi\}$. Set $F(1) = 0$, what do you get for $F(z)$ with $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$?

   Hint: For the second part, find a suitable path joining 1 and $z = r\,\mathrm{e}^{i\varphi}$.

   Remark: This anti-derivative with $F(1) = 0$ is called the *principal value of the (complex) logarithm* and denoted $\mathrm{Log}(z)$.

2.) Formulate and prove the homotopy version of Cauchy's theorem for multiple simple closed contours.
   Hint: Use a drawing for the proof – is this a proof then?

3.) Compute $4 \cdot \int_\Gamma \frac{(1-i)z-(1+i)}{z^3-3z^2-z+3}\,\mathrm{d}z$, where

$$\Gamma = \left\{ z = x + iy \in \mathbb{C} \mid \left(\frac{x}{2}\right)^2 + \left(\frac{y}{7}\right)^2 = 1 \right\}$$

   (an ellipse with half-axes radii 2 and 7).

4.) Prove Gauss' Fundamental Theorem of Algebra.
   Hint: Suppose that a polynomial $p(z)$ has no zeroes and conclude that $f(z) = \frac{1}{p(z)}$ is bounded.)

Optional question:

5.) We again explore *winding numbers* (compare Exercise sheet 4 Question 3),

   (i) Let $\gamma : [a, b] \to \mathbb{C}$ be a simple closed regular path, so that $\Gamma = \gamma([a, b])$ is oriented counter-clockwise as usual. Prove that, for $z \notin \Gamma$,

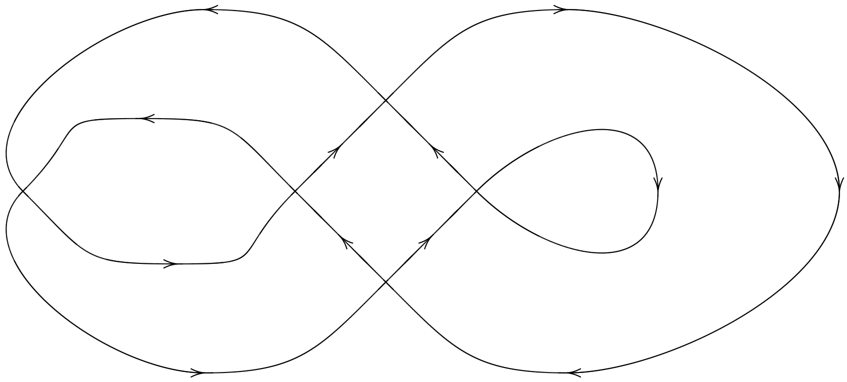
$$w(\gamma, z) = \begin{cases} 1 & \text{if } z \in I_\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Please turn over!

Deduce that, if a domain $D \subset \mathbb{C}$ is simply connected, then $w(\gamma, z) = 0$ for all $z \notin D$ and simple closed regular paths $\gamma : [a, b] \to D$ (compare [ST, Section 8.7]).

Note: This turns out to be also a sufficient condition but this is rather hard to prove in our setup.

(ii) "Compute" the winding numbers for all points $z \in \mathbb{C} \setminus \Gamma$ for the following (oriented) contour "by inspection" (and/or "educated guessing"):



6.) Invent an exam question!