MA30056: Complex Analysis

Exercise Sheet 6: Homotopy version of Cauchy's Theorem and Cauchy Formulae

Please hand solutions in at the lecture on Monday 9th March.

Note: There will be no drop-in session on Tuesday 3rd March.

1.) Prove that the function $f(z) = \frac{1}{z}$ has an anti-derivative F in the *cut plane* $\mathbb{C}\setminus\mathbb{R}_{\leq 0} = \{z = r\, e^{i\varphi}\, | r > 0, -\pi < \varphi < \pi\}$. Set $F(1) = 0$, what do you get for $F(z)$ with $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$?

Hint: For the second part, find a suitable path joining 1 and $z = r\, e^{i\varphi}$.

Remark: This anti-derivative with $F(1) = 0$ is called the *principal value of the (complex) logarithm* and denoted $\mathrm{Log}(z)$.

Solution: We first note that $f(z) = \frac{1}{z}$ is holomorphic in $\mathbb{C}\setminus\{0\}$ (with $f'(z) = -\frac{1}{z^2}$, see Theorem II.2.2).

We now show that the cut plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0} = \{z \in \mathbb{C}\,|\,|z| > 0 \text{ and } \arg(z) \in (-\pi, \pi)\}$ is a star-domain: Let z_0 be any positive real number (e.g., $z_0 = 1$). We show that z_0 centre of the star-domain $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, i.e., that the straight line $[z_0, z] \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ for any $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

- Case $z \in \mathbb{R}_{>0}$: In this case, $[z_0, z]$ is an interval in the positive real numbers.
- Case $\mathrm{Im}\, z \neq 0$: Write $z = x + iy$. Then

$$\begin{aligned}[z_0, z] &= \{z_0 + t(z - z_0)\,|\,0 \leq t \leq 1\} = \{z_0 + t(x - z_0) + i\,t\,y\,|\,0 \leq t \leq 1\} \\ &= \{z_0\} \cup \{z_0 + t(x - z_0) + i\,t\,y\,|\,0 < t \leq 1\} \\ &\subset \{z_0\} \cup (\mathbb{C} \setminus \mathbb{R}) \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}.\end{aligned}$$

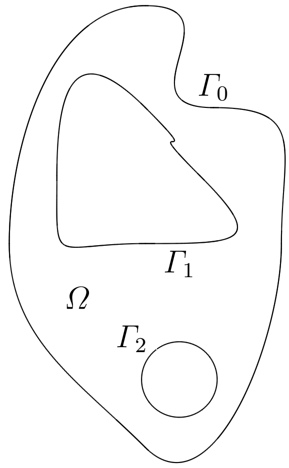
Thus, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is a star-domain, and Corollary III.2.9 ensures the existence of an anti-derivative $F : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}$ of $f(z) = \frac{1}{z}$.

To explicitly calculate the value of the anti-derivative $F(z)$, we use the following path γ joining 1 and $z = r\, e^{i\varphi}$: $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1(t) = 1 + t\,(r - 1)$ and $\gamma_2(t) = r\, e^{i\varphi\, t}$ (in both cases $t \in [0, 1]$). We have

$$\begin{aligned}F(z) &\overset{F(1)=0}{=} \int_\gamma f\, dz = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} = \int_0^1 \frac{r - 1}{1 + t\,(r - 1)}\, dt + \int_0^1 \frac{r\, i\varphi\, e^{i\varphi\, t}}{r\, e^{i\varphi\, t}}\, dt \\ &= \log(r) + i\varphi \;=\; \log(|z|) + i \arg(z).\end{aligned}$$

Note that the integral over γ_1 yields a real integral (and log is the well-known (real) logarithm here).

2.) Formulate and prove the homotopy version of Cauchy's theorem for multiple simple closed contours.
Hint: Use a drawing for the proof – is this a proof then?



Solution: Let $f : D \to \mathbb{C}$ be holomorphic and $\Omega \subset D$ bounded with piecewise regular boundary components $\Gamma_0, \dots \Gamma_n \subset D$ such that $\Omega \subset I_{\Gamma_0}$ is in the interior of Γ_0. Then

$$\int_{\Gamma_0} f \, \mathrm{d}z = \sum_{k=1}^{n} \int_{\Gamma_k} f \, \mathrm{d}z.$$

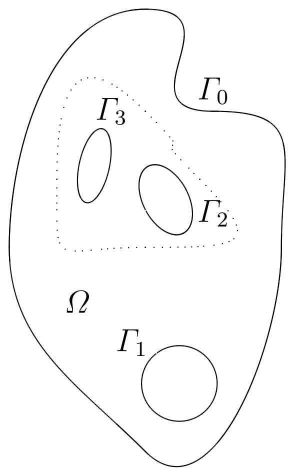
To prove this general Homotopy version of Cauchy's theorem "join" the ("inner") contours Γ_k and Γ_{k+1} by piecewise regular paths α_k, $k = 1, \dots, n-1$ (think about this for a moment: Is this always possible, no matter how many and how complicated the Γ_k are?). In this way we obtain a closed piecewise regular path $\alpha : [a, b] \to D$ (e.g., in the above picture, α consists of Γ_1 joined by α_1 to Γ_2 which is joined by $-\alpha_1$ to Γ_1 again) with

$$\sum_{k=1}^{n} \int_{\Gamma_k} f \, \mathrm{d}z = \int_{\alpha} f \, \mathrm{d}z = \int_{\Gamma_0} f \, \mathrm{d}z,$$

where the second inequality follows from the simple Homotopy version of Cauchy's theorem.

If you insist on requiring the "inner" path α to be simple then the construction becomes slightly more complicated: we also need paths $\tilde{\alpha}_k$ for the "return way".

Using induction reduces the problem to considering two "inner contours", compare the figure to right: If we have three "inner contours", we enclose two of them by an "auxiliary contour" (dotted in the figure to the right) and apply the claim first to Γ_0 with inner contours Γ_1 and the auxiliary contour and then in a second step to the auxiliary contour with inner contours Γ_2 and Γ_3. Continue like this for $n+1$ "inner contours" (e.g., by enclosing n of the contours by an auxiliary contour).



We now prove the claim for two "inner contours": Let $I = I_{\Gamma_0} \setminus (\overline{I}_{\Gamma_1} \cup \overline{I}_{\Gamma_2})$ denote our "working domain". Thus choose $\alpha_1 : [0, 1] \to I \cup \Gamma_1 \cup \Gamma_2$ and then $\tilde{\alpha}_1$ "close by" but disjoint (use that $I \setminus \alpha([0, 1])$ is still connected) and use Cauchy's theorem to argue that the integrals $\int_{\alpha_1} f \, \mathrm{d}z$ and $\int_{\tilde{\alpha}_1} f \, \mathrm{d}z$ cancel.

3.) Compute $4 \cdot \int_\Gamma \frac{(1-i)z-(1+i)}{z^3-3z^2-z+3}\,\mathrm{d}z$, where

$$\Gamma = \left\{ z = x + iy \in \mathbb{C} \mid \left(\frac{x}{2}\right)^2 + \left(\frac{y}{7}\right)^2 = 1 \right\}$$

(an ellipse with half-axes radii 2 and 7).

Solution: First note that

$$4\frac{(1-i)z-(1+i)}{z^3-3z^2-z+3} = \frac{1-2i}{z-3} + \frac{2i}{z-1} - \frac{1}{z+1}.$$

Now, by Cauchy's Theorem, the previous question and the example on p. 39 (Section III.1) in the lecture notes, we have

$$\begin{aligned} 4\int_\Gamma \frac{(1-i)z-(1+i)}{z^3-3z^2-z+3}\,\mathrm{d}z &= \int_{|z-1|=\frac{1}{2}} \frac{2i\,\mathrm{d}z}{z-1} - \int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{z+1} \\ &= -4\pi - 2\pi i = -2(2+i)\pi. \end{aligned}$$

Alternative: We can also use Cauchy's Formula (and don't have to calculate the partial fraction). First, we again use the homotopy version of Cauchy's theorem for multiple simple closed contours (previous question):

$$\begin{aligned} 4\int_\Gamma \frac{(1-i)z-(1+i)}{z^3-3z^2-z+3}\,\mathrm{d}z &= 4\int_{|z-1|=\frac{1}{2}} \frac{((1-i)z-(1+i))\,/\,((z+1)(z-3))}{z-1}\,\mathrm{d}z \\ &\quad + 4\int_{|z+1|=\frac{1}{2}} \frac{((1-i)z-(1+i))\,/\,((z-1)(z-3))}{z+1}\,\mathrm{d}z \\ &= 4\cdot 2\pi i \cdot \frac{(1-i)\cdot 1-(1+i)}{(1+1)(1-3)} \\ &\quad + 4\cdot 2\pi i \cdot \frac{(1-i)\cdot(-1)-(1+i)}{(-1-1)(-1-3)} \\ &= -4\pi - 2\pi i = -2(2+i)\pi. \end{aligned}$$

4.) Prove Gauss' Fundamental Theorem of Algebra.
Hint: Suppose that a polynomial $p(z)$ has no zeroes and conclude that $f(z) = \frac{1}{p(z)}$ is bounded.)

Solution: We proof this by contraposition, i.e., since we want to proof that a nonconstant polynomial has at least one zero, we actually show that a polynomial without a zero is constant.

Suppose the polynomial $p(z) = \sum_{k=0}^n a_k z^k$, $a_n \neq 0$, has no zeroes. Then $f(z) = \frac{1}{p(z)}$ defines an entire function.

Please turn over!

We wish to show that f is bounded:

(i) Since $\frac{p(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n} \to 1$ as $|z| \to \infty$, there is $R > 0$ so that

$$|z| > R \qquad \Rightarrow \qquad \left|\frac{p(z)}{a_n z^n}\right| \geq \frac{1}{2}.$$

Therefore, we have $R > 0$ so that

$$|z| > R \qquad \Rightarrow \qquad |f(z)| \leq \frac{2}{|a_n z^n|} \leq \frac{2}{|a_n|\, R^n}.$$

(ii) Since $\overline{B}_R(0)$ is compact and f is continuous, there is $M \in \mathbb{R}$ so that

$$|z| \leq R \qquad \Rightarrow \qquad |f(z)| \leq M.$$

Thus f is bounded (by $\max\{\frac{2}{|a_n|\,R^n}, M\}$). Hence, by Liouville's Theorem, f is constant – and so is p.

Optional question:

5.) We again explore *winding numbers* (compare Exercise sheet 4 Question 3),

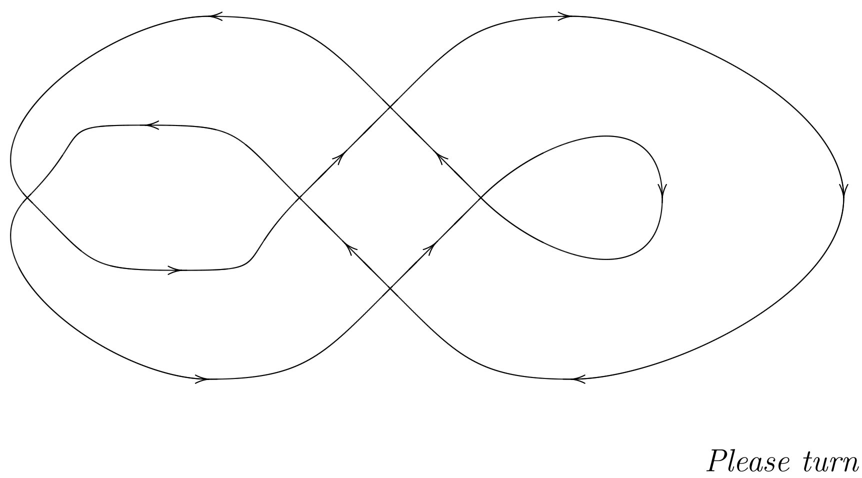
(i) Let $\gamma : [a, b] \to \mathbb{C}$ be a simple closed regular path, so that $\Gamma = \gamma([a, b])$ is oriented counter-clockwise as usual. Prove that, for $z \notin \Gamma$,

$$w(\gamma, z) = \begin{cases} 1 & \text{if } z \in I_\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that, if a domain $D \subset \mathbb{C}$ is simply connected, then $w(\gamma, z) = 0$ for all $z \notin D$ and simple closed regular paths $\gamma : [a, b] \to D$ (compare [ST, Section 8.7]).

Note: This turns out to be also a sufficient condition but this is rather hard to prove in our setup.

(ii) "Compute" the winding numbers for all points $z \in \mathbb{C} \setminus \Gamma$ for the following (oriented) contour "by inspection" (and/or "educated guessing"):



Please turn over!

Solution:

(i) If $z \notin I_\Gamma$ then $\zeta \mapsto \frac{1}{\zeta - z}$ is holomorphic in I_Γ and $w(\gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}\zeta}{\zeta - z} = 0$ by Cauchy's Theorem.

If $z \in I_\Gamma$ then, by the Homotopy version of Cauchy's Theorem,

$$w(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{|\zeta - z| = r} \frac{\mathrm{d}\zeta}{\zeta - z} = 1,$$

where $r > 0$ is chosen so that $\overline{B}_r(z) \subset I_\Gamma$.

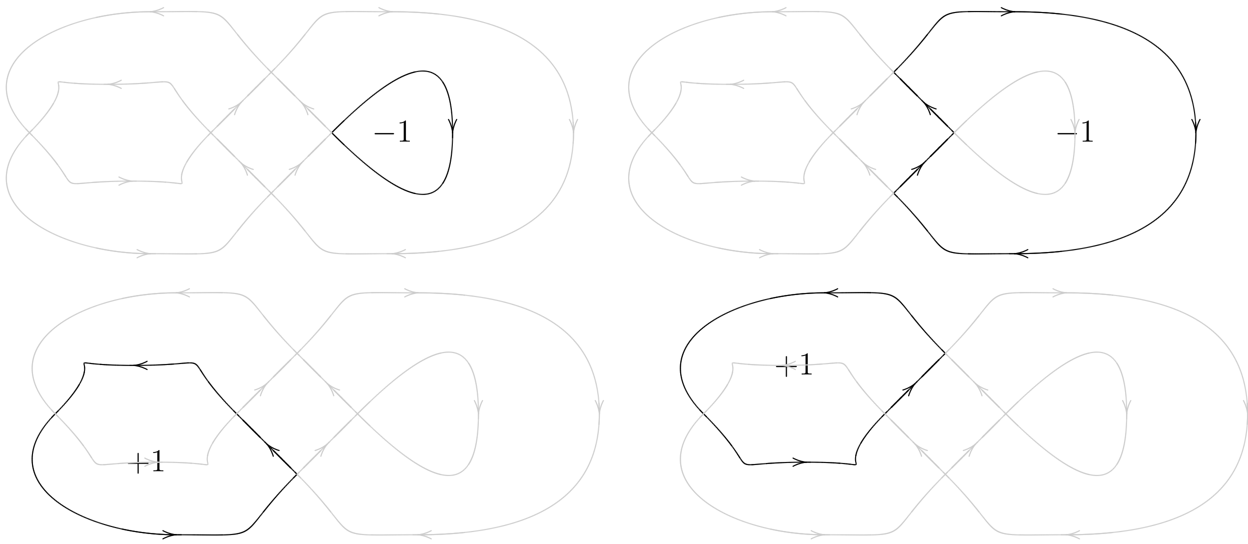
Now, if $D \subset \mathbb{C}$ is simply connected then $I_\Gamma \subset D$ for every simple closed contour $\Gamma \subset D$.

Thus if $\gamma : [a, b] \to D$ is a simple closed regular path, $\Gamma = \gamma([a, b])$, and $z \notin D$ then $z \notin I_\Gamma$ and, therefore, $w(\gamma, z) = 0$.

(ii) We know the following things about winding numbers:

- By Question 3 on Exercise sheet 4, $w(\gamma, z)$ is constant on each (path!) connected set $A \subset \mathbb{C} \setminus \gamma([a, b])$.
- By part (i), if $\gamma : [a, b] \to \mathbb{C}$ is a simple closed regular path that is oriented counter-clockwise, then $w(\gamma, z) = 1$ if $z \in I_\Gamma$ and $w(\gamma, z) = 0$ otherwise.

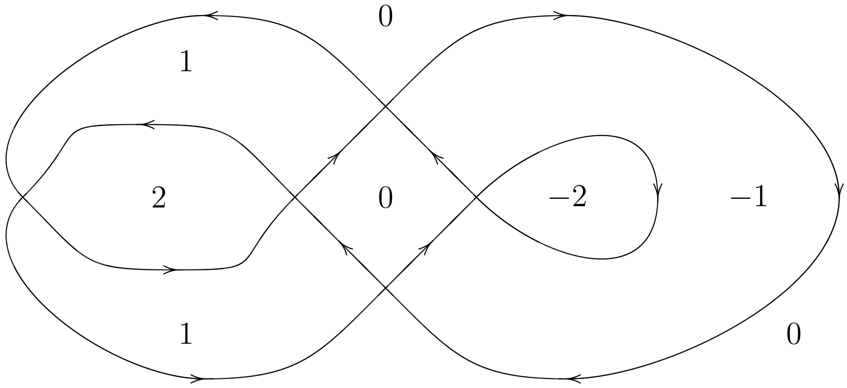
We now break up the (non-simple) path Γ in question into four[1] simple closed piecewise regular paths and then use path-additivity of the integral to obtain the winding number in each path connected set of $\mathbb{C} \setminus \Gamma$. Note that the winding number in the interior of a simple closed (piecewise) regular path is $+1$ if the orientation of the path is counter-clockwise and -1 if the orientation of the path is clockwise.



Please turn over!

[1]Sorry, for some strange reason the picture of the contour does not scale down "faithfully".

Putting everything together yields:



Remark: For more on how to compute winding numbers, see [I. Stewart & D. Tall: Complex Analysis; Sections 7.7 & 7.8].

6.) Invent an exam question!