MA30056: Complex Analysis

Exercise Sheet 5: Cauchy's Theorem

Please hand solutions in at the lecture on Monday 2nd March.

Note: There will be no drop-in session on Tuesday 3rd March.

1.) Let $\Gamma \subset \mathbb{C}$ be a simple closed contour and $z_0 \in I_\Gamma$ a point in its interior. Prove that there is a $\varrho > 0$ so that
$$\left| \int_\Gamma \frac{\mathrm{d}z}{(z - z_0)^k} \right| < \frac{1}{\varrho^k}\, L,$$
where $k \in \mathbb{N}$ and L is the length of Γ.

2.) Verify that the complex path integral can be written in terms of two real path integrals
$$\int_\gamma f \,\mathrm{d}z = \int_\gamma u \;\mathrm{d}x - v \;\mathrm{d}y + i \int_\gamma v \;\mathrm{d}x + u \;\mathrm{d}y,$$
where $f = u + iv$.

3.) Use the previous question and Green's Theorem to prove the following weak version of Cauchy's Theorem:

Theorem III.2.7. Let $f : D \to \mathbb{C}$ be holomorphic in a domain D, with continuous f', and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Then $\int_\Gamma f \,\mathrm{d}z = 0$.

Hint: **Green's Theorem.** Let $\alpha, \beta : \mathbb{R}^2 \supset D \to \mathbb{R}$ be continuously differentiable and $\Omega \subset D$ bounded with piecewise smooth boundary $\partial\Omega$. Then
$$\int_{\partial\Omega} \alpha \;\mathrm{d}x + \beta \;\mathrm{d}y = \iint_\Omega (\beta_x - \alpha_y) \;\mathrm{d}x \,\mathrm{d}y,$$
where the boundary integral is taken in the anti-clockwise sense. □

4.) *will be on next week's sheet*

Please turn over!

Optional question:

5.) Fill in the details (all "__________") in the following proof of Lemma III.2.6.

Lemma III.2.6. Let $f(z,w)$ and[1] $f_z(z,w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z,w)\,\mathrm{d}w$ is holomorphic in D, and

$$F'(z) = \int_\Gamma f_z(z,w)\,\mathrm{d}w.$$

Proof. Let $F(z) = U(x,y) + iV(x,y)$ and $f(z,w) = u\,(x,y,\xi(t),\eta(t)) + iv\,(x,y,\xi(t),\eta(t))$, where $\gamma(t) = \xi(t) + i\eta(t)$ is a parametrisation of Γ with $t \in [a,b]$. Then

$$U(x,y) = \int_a^b \underline{\hspace{2cm}} \qquad \text{and} \qquad V(x,y) = \int_a^b \underline{\hspace{2cm}}.$$

If $z_0 = x_0 + iy_0 \in D$, then (here $h \in \mathbb{R}$)

$$\begin{aligned} U_x(x_0,y_0) = \lim_{h\to 0} \frac{U(x_0+h,y_0) - U(x_0,y_0)}{h} \quad &= \quad \lim_{h\to 0} \frac{1}{h} \int_a^b \underline{\hspace{2cm}} \\ \overset{\text{MVT}}{=} \lim_{h\to 0} \int_a^b \underline{\hspace{2cm}} \quad &= \quad \int_a^b \left[u_x(x_0,y_0,\xi,\eta)\,\xi' - v_x(x_0,y_0,\xi,\eta)\,\eta'\right] \mathrm{d}t. \end{aligned}$$

The justification for taking the limit under the integral sign in the last step is supplied by __________ and the following lemma (for a proof see [DET, Theorem 3.5.2]):

> **Lemma.** Let $f(z,w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z,w)\,\mathrm{d}w$ is continuous in D. □

Similarly,

$$U_y(x_0,y_0) = \int_a^b \underline{\hspace{2cm}}, \qquad V_x(x_0,y_0) = \int_a^b \underline{\hspace{2cm}},$$

$$V_y(x_0,y_0) = \int_a^b \underline{\hspace{2cm}}.$$

Since $f(z,w)$ is holomorphic at z_0, we have $u_x(x_0,y_0,\xi,\eta) = v_y(x_0,y_0,\xi,\eta)$ and __________=__________, and therefore

$$U_x(x_0,y_0) = \underline{\hspace{2cm}} \qquad \text{and} \qquad \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

Hence, $F(z)$ is holomorphic at z_0 and

$$F'(z_0) = U_x(x_0,y_0) + i\,V_x(x_0,y_0) = \int_a^b \underline{\hspace{2cm}} = \int_\Gamma \underline{\hspace{2cm}}.$$

Since z_0 is any point in D, the result holds throughout D. □

[1] So, $f(z,w)$ is complex differentiable with respect to z, i.e.,

$$f_z(z_0,w) = \lim_{z\to z_0} \frac{f(z,w) - f(z_0,w)}{z - z_0}.$$