MA30056: Complex Analysis

Exercise Sheet 5: Cauchy's Theorem

Please hand solutions in at the lecture on Monday 2nd March.

Note: There will be no drop-in session on Tuesday 3rd March.

1.) Let $\Gamma \subset \mathbb{C}$ be a simple closed contour and $z_0 \in I_\Gamma$ a point in its interior. Prove that there is a $\varrho > 0$ so that

$$\left| \int_\Gamma \frac{\mathrm{d}z}{(z-z_0)^k} \right| < \frac{1}{\varrho^k} L,$$

where $k \in \mathbb{N}$ and L is the length of Γ.

Solution: This is an application of the ML-inequality: We have

$$M = \max \left\{ \left| \frac{1}{(z-z_0)^k} \right| \mid z \in \Gamma \right\} = \max \left\{ \left| \frac{1}{z-z_0} \right|^k \mid z \in \Gamma \right\}.$$

Thus setting $\varrho = \min\{|z-z_0| \mid z \in \Gamma\}$ (note that $\varrho > 0$ since $z_0 \in I_\Gamma$, and that this minimum is attained since Γ is compact), we obtain the estimate $M \leq \frac{1}{\varrho^k}$ and thus the required result.

Note: Note that you cannot use Cauchy's theorem here, since $\frac{1}{(z-z_0)^k}$ is not holomorphic in I_Γ. However, using the homotopy version of Cauchy's theorem and the formulae for $\int_{\partial B_r(z_0)} \frac{\mathrm{d}z}{(z-z_0)^k}$ we see that most of these integrals vanish.

2.) Verify that the complex path integral can be written in terms of two real path integrals

$$\int_\gamma f \,\mathrm{d}z = \int_\gamma u \,\mathrm{d}x - v \,\mathrm{d}y + i \int_\gamma v \,\mathrm{d}x + u \,\mathrm{d}y,$$

where $f = u + iv$.

Solution: The short (and sloppy) solution:

$$f \,\mathrm{d}z = (u + i\,v)(\mathrm{d}x + i\,\mathrm{d}y) = (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\,(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

The (slightly) longer solution: write $\gamma = (\alpha, \beta)$; then (we suppress the argument t)

$$\begin{aligned} f(\gamma)\,\gamma' &= (u(\alpha,\beta) + iv(\alpha,\beta))\,(\alpha' + i\beta') \\ &= (u(\alpha,\beta)\alpha' - v(\alpha,\beta)\beta') + i\,(v(\alpha,\beta)\alpha' + u(\alpha,\beta)\beta'), \end{aligned}$$

which gives the claim when writing the real line integrals out (e.g., $\int_\gamma u\,\mathrm{d}x = \int u(\alpha,\beta)\alpha'\,\mathrm{d}t$).

3.) Use the previous question and Green's Theorem to prove the following weak version of Cauchy's Theorem:

Theorem III.2.7. Let $f : D \to \mathbb{C}$ be holomorphic in a domain D, with continuous f', and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Then $\int_\Gamma f \,\mathrm{d}z = 0$.

Hint: **Green's Theorem.** Let $\alpha, \beta : \mathbb{R}^2 \supset D \to \mathbb{R}$ be continuously differentiable and $\Omega \subset D$ bounded with piecewise smooth boundary $\partial\Omega$. Then

$$\int_{\partial\Omega} \alpha \,\mathrm{d}x + \beta \,\mathrm{d}y = \iint_\Omega (\beta_x - \alpha_y) \,\mathrm{d}x \,\mathrm{d}y,$$

where the boundary integral is taken in the anti-clockwise sense. □

Solution: By the Jordan Curve Thmeorem, $\Gamma = \partial I_\Gamma$ is the boundary of its (bounded) interior I_Γ so that we can apply Green's Theorem.

Writing $f = u + iv$ we compute

$$\begin{aligned}\int_\Gamma f \,\mathrm{d}z &= \int_{\partial I_\Gamma} u \,\mathrm{d}x - v \,\mathrm{d}y + i \int_{\partial I_\Gamma} v \,\mathrm{d}x + u \,\mathrm{d}y \\ &= \iint_{I_\Gamma} -(v_x + u_y) \,\mathrm{d}x \,\mathrm{d}y + i \iint_{I_\Gamma} (u_x - v_y) \,\mathrm{d}x \,\mathrm{d}y,\end{aligned}$$

by Green's Theorem, so that the claim follows from the Cauchy-Riemann equations.

4.) *will be on next week's sheet*

Optional question:

5.) Fill in the details (all "__________") in the following proof of Lemma III.2.6.

Lemma III.2.6. Let $f(z,w)$ and[1] $f_z(z,w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z,w) \,\mathrm{d}w$ is holomorphic in D, and

$$F'(z) = \int_\Gamma f_z(z,w) \,\mathrm{d}w.$$

Proof. Let $F(z) = U(x,y) + iV(x,y)$ and $f(z,w) = u\,(x,y,\xi(t),\eta(t)) + iv\,(x,y,\xi(t),\eta(t))$, where $\gamma(t) = \xi(t) + i\eta(t)$ is a parametrisation of Γ with $t \in [a,b]$. Then

$$U(x,y) = \int_a^b \underline{\qquad\qquad} \qquad \text{and} \qquad V(x,y) = \int_a^b \underline{\qquad\qquad}.$$

Please turn over!

[1] So, $f(z,w)$ is complex differentiable with respect to z, i.e.,

$$f_z(z_0,w) = \lim_{z \to z_0} \frac{f(z,w) - f(z_0,w)}{z - z_0}.$$

If $z_0 = x_0 + iy_0 \in D$, then (here $h \in \mathbb{R}$)

$$U_x(x_0, y_0) = \lim_{h\to 0} \frac{U(x_0+h, y_0) - U(x_0, y_0)}{h} \quad = \quad \lim_{h\to 0} \frac{1}{h} \int_a^b \underline{\qquad\qquad\qquad}$$

$$\stackrel{\text{MVT}}{=} \lim_{h\to 0} \int_a^b \underline{\qquad\qquad\qquad} \quad = \quad \int_a^b [u_x(x_0, y_0, \xi, \eta)\, \xi' - v_x(x_0, y_0, \xi, \eta)\, \eta']\, \mathrm{d}t.$$

The justification for taking the limit under the integral sign in the last step is supplied by __________ and the following lemma (for a proof see [DET, Theorem 3.5.2]):

> **Lemma.** Let $f(z, w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z, w)\, \mathrm{d}w$ is continuous in D. □

Similarly,

$$U_y(x_0, y_0) = \int_a^b \underline{\qquad\qquad\qquad}, \qquad\qquad V_x(x_0, y_0) = \int_a^b \underline{\qquad\qquad\qquad},$$

$$V_y(x_0, y_0) = \int_a^b \underline{\qquad\qquad\qquad}.$$

Since $f(z, w)$ is holomorphic at z_0, we have $u_x(x_0, y_0, \xi, \eta) = v_y(x_0, y_0, \xi, \eta)$ and __________=__________, and therefore

$$U_x(x_0, y_0) = \underline{\qquad\qquad\qquad} \qquad \text{and} \qquad \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad}.$$

Hence, $F(z)$ is holomorphic at z_0 and

$$F'(z_0) = U_x(x_0, y_0) + i\, V_x(x_0, y_0) = \int_a^b \underline{\qquad\qquad\qquad} = \int_\Gamma \underline{\qquad\qquad\qquad}.$$

Since z_0 is any point in D, the result holds throughout D. □

Solution:

Proof. Let $F(z) = U(x, y) + iV(x, y)$ and $f(z, w) = u\,(x, y, \xi(t), \eta(t)) + iv\,(x, y, \xi(t), \eta(t))$, where $\gamma(t) = \xi(t) + i\eta(t)$ is a parametrisation of Γ with $t \in [a, b]$. Then (compare the following with Question 2)

$$U(x, y) = \int_a^b (u\,\xi' - v\,\eta')\, \mathrm{d}t \qquad \text{and} \qquad V(x, y) = \int_a^b (u\,\eta' + v\,\xi')\, \mathrm{d}t.$$

If $z_0 = x_0 + iy_0 \in D$, then (here $h \in \mathbb{R}$)

$$\begin{aligned}
U_x(x_0, y_0) &\stackrel{\text{def. part. der.}}{=} \lim_{h\to 0} \frac{U(x_0+h, y_0) - U(x_0, y_0)}{h} \\
&\stackrel{\text{def. U}}{=} \lim_{h\to 0} \frac{1}{h} \int_a^b [u(x_0+h, y_0, \xi, \eta)\, \xi' - v(x_0+h, y_0, \xi, \eta)\, \eta' \\
&\qquad\qquad -u(x_0, y_0, \xi, \eta)\, \xi' + v(x_0, y_0, \xi, \eta)\, \eta']\, \mathrm{d}t \\
&\stackrel{\text{MVT}}{=} \lim_{h\to 0} \int_a^b [u_x(x_0+\vartheta_1 h, y_0, \xi, \eta)\, \xi' - v_x(x_0+\vartheta_2 h, y_0, \xi, \eta)\, \eta']\, \mathrm{d}t \\
&= \int_a^b [u_x(x_0, y_0, \xi, \eta)\, \xi' - v_x(x_0, y_0, \xi, \eta)\, \eta']\, \mathrm{d}t.
\end{aligned}$$

Please turn over!

The justification for taking the limit under the integral sign is supplied by the continuity of the first partial derivatives and the following lemma (for a proof see [DET,Theorem 3.5.2]):

> **Lemma.** Let $f(z,w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z,w)\,\mathrm{d}w$ is continuous in D. □

Similarly,

$$U_y(x_0,y_0) = \int_a^b [u_y(x_0,y_0,\xi,\eta)\,\xi' - v_y(x_0,y_0,\xi,\eta)\,\eta']\,\mathrm{d}t,$$
$$V_x(x_0,y_0) = \int_a^b [u_x(x_0,y_0,\xi,\eta)\,\eta' + v_x(x_0,y_0,\xi,\eta)\,\xi']\,\mathrm{d}t,$$
$$V_y(x_0,y_0) = \int_a^b [u_y(x_0,y_0,\xi,\eta)\,\eta' + v_y(x_0,y_0,\xi,\eta)\,\xi']\,\mathrm{d}t.$$

Since $f(z,w)$ is holomorphic at z_0 (w.r.t. z), we have $u_x(x_0,y_0,\xi,\eta) = v_y(x_0,y_0,\xi,\eta)$ and $u_y(x_0,y_0,\xi,\eta) = -v_x(x_0,y_0,\xi,\eta)$, and therefore

$$U_x(x_0,y_0) = V_y(x_0,y_0) \qquad \text{and} \qquad U_y(x_0,y_0) = -V_x(x_0,y_0).$$

Hence, $F(z)$ is holomorphic at z_0 (by the sufficient Cauchy-Riemann conditions) and

$$F'(z_0) = U_x(x_0,y_0) + i\,V_x(x_0,y_0) = \int_a^b (u_x + iv_x)(\xi' + i\eta')\,\mathrm{d}t = \int_\Gamma f_z(z_0,w)\,\mathrm{d}w.$$

Since z_0 is any point in D, the result holds throughout D. □