MA30056: Complex Analysis

Exercise Sheet 4: Path integrals

Please hand solutions in at the lecture on Monday 23rd February.

1.) Compute the path integrals $\int_{\gamma_i} f_j \, \mathrm{d}z$ for

$$f_1(z) = z, \qquad f_2(z) = \overline{z}, \qquad \text{and} \qquad f_3(z) = |z|^2,$$

where

- γ_1 is the straight line segment from $z = 0$ to $z = 1 + i$, and
- γ_2 is the polygonal path from $z = 0$ to $z = 1 + i$ via $z = 1$.

What do you observe?

2.) We call two regular paths $\gamma : [a, b] \to \mathbb{C}$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ *equivalent*, $\tilde{\gamma} \sim \gamma$, if there is a regular and onto $h : [\tilde{a}, \tilde{b}] \to [a, b]$ with $\tilde{\gamma} = \gamma \circ h$ so that $h' > 0$. Prove that $\sim$ is an equivalence relation.

3.) Let $\gamma : [a, b] \to \mathbb{C}$ be a closed regular path. For $z \notin \gamma([a, b])$ we define the *winding number* of γ around z by

$$w(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Prove that

(i) $w(\gamma, z) \in \mathbb{Z}$
Hint: write $\varphi(t) = \int_a^t \frac{\gamma'(\tau)\,\mathrm{d}\tau}{\gamma(\tau) - z}$ and show that $(\gamma(t) - z)\,\mathrm{e}^{-\varphi(t)}$ is constant – remember that $\mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y)$.

(ii) $z \mapsto w(\gamma, z)$ is constant on each (path!) connected set $A \subset \mathbb{C} \setminus \gamma([a, b])$
Hint: first prove that $z \mapsto w(\gamma, z)$ is continuous by using the ML-inequality.

4.) Prove that the function $f(z) = \frac{1}{z}$ does not have an anti-derivative in any punctured neighbourhood of the origin $z = 0$ (e.g., $B_\varepsilon(0) \setminus \{0\}$).

Please turn over!

Optional question:

5.) Recall that for a continuous vector field $\vec{v} = (v_1, v_2) : D \to \mathbb{R}^2$ and a smooth path $\gamma : [a, b] \to D$, $\gamma(t) = (x(t), y(t))$ the line integral is defined by

$$\int_\gamma \vec{v} \cdot \mathrm{d}\vec{s} = \int_\gamma v_1 \;\mathrm{d}x + v_2 \;\mathrm{d}y = \int_a^b \vec{v}(\gamma(t)) \cdot \gamma'(t) \;\mathrm{d}t$$

One can show (we take it as a definition here!) that for a simple closed path γ

$$F(\gamma) = \int_\gamma x \;\mathrm{d}y$$

is the area enclosed by γ (i.e., the area of I_γ). How can we interpret the path integral $\int_\gamma \overline{z} \;\mathrm{d}z$ for a simple closed smooth path γ in $\mathbb{C}$?