MA30056: Complex Analysis

Exercise Sheet 4: Path integrals

Please hand solutions in at the lecture on Monday 23rd February.

1.) Compute the path integrals $\int_{\gamma_i} f_j \, \mathrm{d}z$ for

$$f_1(z) = z, \qquad f_2(z) = \overline{z}, \qquad \text{and} \qquad f_3(z) = |z|^2,$$

where

- γ_1 is the straight line segment from $z = 0$ to $z = 1 + i$, and
- γ_2 is the polygonal path from $z = 0$ to $z = 1 + i$ via $z = 1$.

What do you observe?

Solution: We parametrize the line segment by $[0, 1] \ni t \mapsto \gamma_1(t) = t\,(1+i)$. Then

$$\int_{\gamma_1} f_1 \, \mathrm{d}z = \int_0^1 t(1+i)^2 \, \mathrm{d}t = \frac{1}{2}(1+i)^2 = i,$$

$$\int_{\gamma_1} f_2 \, \mathrm{d}z = \int_0^1 t|1+i|^2 \, \mathrm{d}t = \frac{1}{2} 2 = 1,$$

$$\int_{\gamma_1} f_3 \, \mathrm{d}z = \int_0^1 t^2|1+i|^2(1+i) \, \mathrm{d}t = \frac{2}{3}(1+i).$$

Next, $\gamma_2 = \gamma_{21} + \gamma_{22}$ with $\gamma_{21}(t) = t$ and $\gamma_{22}(t) = 1 + ti$, where $t \in [0, 1]$ for both regular pieces (note that we may choose the parameter domain at our convenience because of the parameter invariance of the path integral). Thus

$$\int_{\gamma_2} f_1 \, \mathrm{d}z = \int_{\gamma_{21}} f_1 \, \mathrm{d}z + \int_{\gamma_{22}} f_1 \, \mathrm{d}z = \int_0^1 t \, \mathrm{d}t + \int_0^1 (1+ti)i \, \mathrm{d}t = \frac{1}{2} + i + \frac{1}{2}i^2 = i,$$

$$\int_{\gamma_2} f_2 \, \mathrm{d}z = \int_{\gamma_{21}} f_2 \, \mathrm{d}z + \int_{\gamma_{22}} f_2 \, \mathrm{d}z = \int_0^1 t \, \mathrm{d}t + \int_0^1 (1-ti)i \, \mathrm{d}t = \frac{1}{2} + i - \frac{1}{2}i^2 = 1 + i,$$

$$\int_{\gamma_2} f_3 \, \mathrm{d}z = \int_{\gamma_{21}} f_3 \, \mathrm{d}z + \int_{\gamma_{22}} f_3 \, \mathrm{d}z = \int_0^1 t^2 \, \mathrm{d}t + \int_0^1 (1+t^2)i \, \mathrm{d}t = \frac{1}{3} + i + \frac{i}{3} = \frac{1+4i}{3}.$$

The interesting observation here is that the first (holomorphic!) function integrates to the same value whereas the integrals of (the non-holomorphic) f_2 and f_3 depend on the path we integrate over.

Please turn over!

Remark: Alternatively, one can calculate the above integrals also via

$$\int_\gamma f\,\mathrm{d}z = \int_a^b U(t)\,\mathrm{d}t + \int_a^b V(t)\,\mathrm{d}t,$$

where $U(t) = \mathrm{Re}\,(f(\gamma(t)) \cdot \gamma'(t))$ and $V(t) = \mathrm{Im}\,(f(\gamma(t)) \cdot \gamma'(t))$ (and with $\gamma : [a,b] \to \mathbb{C}$), or via[1]

$$\int_\gamma f\,\mathrm{d}z = \int_\gamma u\ \mathrm{d}x - v\ \mathrm{d}y + i\int_\gamma v\ \mathrm{d}x + u\ \mathrm{d}y,$$

where $f = u + iv$ (see next exercise sheet).

2.) We call two regular paths $\gamma : [a,b] \to \mathbb{C}$ and $\tilde{\gamma} : [\tilde{a},\tilde{b}] \to \mathbb{C}$ *equivalent*, $\tilde{\gamma} \sim \gamma$, if there is a regular and onto $h : [\tilde{a},\tilde{b}] \to [a,b]$ with $\tilde{\gamma} = \gamma \circ h$ so that $h' > 0$. Prove that $\sim$ is an equivalence relation.

Solution: A relation $\sim$ on a set X is called an equivalence relation if it is

(i) "reflexive", i.e., $x \sim x$ for any $x \in X$;

(ii) "symmetric", i.e., $x \sim y \Rightarrow y \sim x$ for $x, y \in X$; and

(iii) "transitive", i.e., $x \sim y$ and $y \sim z$ implies $x \sim z$ for $x, y, z \in X$.

Here, X is the set of regular paths. To verify that $\sim$ is an equivalence relation is straightforward from the definition of $\sim$.

(i) Reflexivity: if $\gamma : [a,b] \to \mathbb{C}$ then $\gamma = \gamma \circ h$ with $h : [a,b] \to [a,b]$, $t \mapsto h(t) = t$. Hence $\gamma \sim \gamma$.

(ii) Symmetry: if $\gamma : [a,b] \to \mathbb{C}$ and $\tilde{\gamma} = \gamma \circ h : [\tilde{a},\tilde{b}] \to \mathbb{C}$ with some regular $h : [\tilde{a},\tilde{b}] \to [a,b]$, then $h^{-1} : [a,b] \to [\tilde{a},\tilde{b}]$ is regular (since $(h^{-1})' = \frac{1}{h'} > 0$) and $\gamma = \gamma \circ h \circ h^{-1} = \tilde{\gamma} \circ h^{-1}$. Hence $\tilde{\gamma} \sim \gamma \Rightarrow \gamma \sim \tilde{\gamma}$.

(iii) Transitivity: if $\gamma : [a,b] \to \mathbb{C}$, $\tilde{\gamma} = \gamma \circ h : [\tilde{a},\tilde{b}] \to \mathbb{C}$ and $\hat{\gamma} = \tilde{\gamma} \circ g : [\hat{a},\hat{b}] \to \mathbb{C}$ with regular $h : [\tilde{a},\tilde{b}] \to [a,b]$ and $g : [\hat{a},\hat{b}] \to [\tilde{a},\tilde{b}]$ then $\hat{\gamma} = \tilde{\gamma} \circ g = (\gamma \circ h) \circ g = \gamma \circ (h \circ g)$ and $h \circ g : [\hat{a},\hat{b}] \to [a,b]$ is regular (chain rule!). Thus $\hat{\gamma} \sim \tilde{\gamma}$ and $\tilde{\gamma} \sim \gamma$ implies $\hat{\gamma} \sim \gamma$.

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[1] e.g., for $f(z) = \overline{z} = x - iy$ and γ_1 one has:

$$\int_{\gamma_1} \overline{z}\,\mathrm{d}z = \int_{\gamma_1} x\,\mathrm{d}x + y\,\mathrm{d}y + i\int_{\gamma_1} (-y)\,\mathrm{d}x + x\,\mathrm{d}y = \frac{1}{2}x^2\Big|_{x,y=0}^{1} + \frac{1}{2}y^2\Big|_{x,y=0}^{1} + i\left(-yx|_{x,y=0}^{1} + xy|_{x,y=0}^{1}\right)$$
$$= \frac{1}{2} + \frac{1}{2} + i\,0 = 1.$$

Remark: Once one has an equivalence relation on some set, one can use that equivalence relation to form “equivalence classes”. In our case this can be used to obtain an alternative approach to orientation.

3.) Let $\gamma : [a, b] \to \mathbb{C}$ be a closed regular path. For $z \notin \gamma([a, b])$ we define the *winding number* of γ around z by

$$w(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Prove that

(i) $w(\gamma, z) \in \mathbb{Z}$
Hint: write $\varphi(t) = \int_a^t \frac{\gamma'(\tau)\,\mathrm{d}\tau}{\gamma(\tau)-z}$ and show that $(\gamma(t) - z)\,\mathrm{e}^{-\varphi(t)}$ is constant – remember that $\mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y)$.

(ii) $z \mapsto w(\gamma, z)$ is constant on each (path!) connected set $A \subset \mathbb{C} \setminus \gamma([a, b])$
Hint: first prove that $z \mapsto w(\gamma, z)$ is continuous by using the ML-inequality.

Solution:

(i) Write $\varphi(t) = \int_a^t \frac{\gamma'(\tau)\,\mathrm{d}\tau}{\gamma(\tau)-z}$. Then (noting that $\varphi'(t) = \gamma'(t)/(\gamma(t) - z)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}((\gamma(t) - z)\,\mathrm{e}^{-\varphi(t)}) = (\gamma'(t) - \varphi'(t)(\gamma(t) - z))\,\mathrm{e}^{-\varphi(t)} = 0,$$

that is, $(\gamma(t) - z)\,\mathrm{e}^{-\varphi(t)}$ is constant. In particular

$$(\gamma(b) - z)\,\mathrm{e}^{-2\pi i\, w(\gamma,z)} = (\gamma(a) - z) \qquad \Leftrightarrow \qquad \mathrm{e}^{-2\pi i\, w(\gamma,z)} = 1$$

since γ is closed, i.e., $\gamma(b) = \gamma(a)$ (and $\gamma(a) - z \neq 0$). Therefore

$$\begin{aligned} \mathrm{e}^{2\pi\, \mathrm{Im}\, w(\gamma,z)} &= 1 \qquad \Leftrightarrow \qquad \mathrm{Im}\, w(\gamma, z) = 0 \\ \mathrm{e}^{-2\pi i\, \mathrm{Re}\, w(\gamma,z)} &= 1 \qquad \Leftrightarrow \qquad 2\pi\, \mathrm{Re}\, w(\gamma, z) = 2\pi k \end{aligned}$$

for some $k \in \mathbb{Z}$.

(ii) Take $z \in \mathbb{C} \setminus \gamma([a, b])$ and $\varrho > 0$ so that $B_{2\varrho}(z) \subset \mathbb{C} \setminus \gamma([a, b])$ ($\mathbb{C} \setminus \gamma([a, b])$ is open since $\gamma([a, b])$ is closed). Then

$$\forall \tilde{z} \in B_\varrho(z)\, \forall \zeta \in \gamma([a, b]) : |\zeta - \tilde{z}| \geq \varrho$$

(so, $\frac{1}{|\zeta-\tilde{z}|} \leq \frac{1}{\varrho}\ \forall \tilde{z} \in B_\varrho(z)$). Hence, by the ML-inequality,

$$\begin{aligned} |w(\gamma, z) - w(\gamma, \tilde{z})| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{z - \tilde{z}}{(\zeta - z)(\zeta - \tilde{z})}\, \mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} \max_{\zeta \in \gamma([a,b])} \left| \frac{z - \tilde{z}}{(\zeta - z)(\zeta - \tilde{z})} \right| L \\ &\leq \frac{L}{2\pi \varrho^2} |z - \tilde{z}| \to 0 \qquad \text{as} \quad \tilde{z} \to z. \end{aligned}$$

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Consequently $\mathbb{C} \setminus \gamma([a,b]) \ni z \mapsto w(\gamma, z) \in \mathbb{Z}$ is continuous (even Lipschitz continuous).

Now, if $z, \tilde{z} \in A$ where $A \subset \mathbb{C} \setminus \gamma([a,b])$ is connected, then there is a path $c : [0,1] \to A$ joining z and $\tilde{z}$, $c(0) = z$ and $c(1) = \tilde{z}$, and the map

$$[0,1] \ni s \mapsto g(s) = w(\gamma, c(s)) \in \mathbb{Z} \subset \mathbb{R}$$

is continuous. If $g(0) \neq g(1)$, say $g(0) < g(1)$, then the IVT would ensure the existence of $s \in (0,1)$ with $g(0) < g(s) = g(0) + \frac{1}{2} < g(1)$ which is not possible since g is $\mathbb{Z}$-valued.

Consequently, g is constant and $w(\gamma, \tilde{z}) = w(\gamma, z)$.

4.) Prove that the function $f(z) = \frac{1}{z}$ does not have an anti-derivative in any punctured neighbourhood of the origin $z = 0$ (e.g., $B_\varepsilon(0) \setminus \{0\}$).

Solution: Suppose, for a contradiction, that $f(z) = \frac{1}{z}$ has an anti-derivative $F : U \to \mathbb{C}$ on a punctured neighbourhood U of the origin. In particular, F is then defined on a punctured disk $B_{2\varepsilon}(0) \setminus \{0\}$ for some $\varepsilon > 0$.

Then we would have $\int_{|z|=\varepsilon} f(z)\, \mathrm{d}z = 0$, in contradiction to $\int_{|z|=\varepsilon} \frac{\mathrm{d}z}{z} = 2\pi i \neq 0$ (as we know from the lecture, see the example on p. 39 in the lecture notes).

Optional question:

5.) Recall that for a continuous vector field $\vec{v} = (v_1, v_2) : D \to \mathbb{R}^2$ and a smooth path $\gamma : [a,b] \to D$, $\gamma(t) = (x(t), y(t))$ the line integral is defined by

$$\int_\gamma \vec{v} \cdot \mathrm{d}\vec{s} = \int_\gamma v_1 \ \mathrm{d}x + v_2 \ \mathrm{d}y = \int_a^b \vec{v}(\gamma(t)) \cdot \gamma'(t) \ \mathrm{d}t$$

One can show (we take it as a definition here!) that for a simple closed path γ

$$F(\gamma) = \int_\gamma x \ \mathrm{d}y$$

is the area enclosed by γ (i.e., the area of I_γ). How can we interpret the path integral $\int_\gamma \overline{z} \ \mathrm{d}z$ for a simple closed smooth path γ in $\mathbb{C}$?

Solution: You might want to look at Question 1 to get a clue: Using the function $f_2(z) = \overline{z}$ and the path $\gamma_2 - \gamma_1$ (which encloses a triangle – half the unit square) we get the value $1 + i - 1 = i$ and therefore $2i$ times the area of the triangle. We

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now check that this in general the case: $\int_\gamma \overline{z}\,\mathrm{d}z$ yields $2i\,F(\gamma)$, i.e., $2i$ times the area of I_Γ:

Using a remark on p. 37 in the lecture notes (also see Question 2 on Exercise sheet 5), we have

$$\int_\gamma \overline{z}\,\mathrm{d}z = \int_\gamma x\,\mathrm{d}x + y\,\mathrm{d}y + i\int_\gamma x\,\mathrm{d}y - y\,\mathrm{d}x \tag{1}$$

A quick way to evaluate these integrals is to use Green's Theorem (compare M10, also see Question 3 on Exercise sheet 5 or p. 45 in the lecture notes) which states:

Green's Theorem. Let $\alpha, \beta : \mathbb{R}^2 \supset D \to \mathbb{R}$ be continuously differentiable and $\Omega \subset D$ bounded with piecewise smooth boundary $\partial\Omega$. Then

$$\int_{\partial\Omega} \alpha\ \mathrm{d}x + \beta\ \mathrm{d}y = \iint_\Omega (\beta_x - \alpha_y)\ \mathrm{d}x\,\mathrm{d}y,$$

where the boundary integral is taken in the anti-clockwise sense. □

Using Green's Theorem, we now immediately get $\int_\gamma \overline{z}\ \mathrm{d}z = 2i \cdot F(\gamma)$. (We even see that $F(\gamma) = \iint_{I_\Gamma} \mathrm{d}x\,\mathrm{d}y$.)

Alternatively, we can observe that the first integral on the right (the real part) in Eq. (1) of course vanishes, since for a closed path we have $x(a) = x(b)$ and $y(a) = y(b)$ and thus

$$\int_\gamma x\,\mathrm{d}x + y\,\mathrm{d}y = \int_a^b (x(t)\,x'(t) + y(t)\,y'(t))\,\mathrm{d}t = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t)\Big|_a^b = 0.$$

Similarly, we have

$$\begin{aligned}\int_\gamma x\,\mathrm{d}y + y\,\mathrm{d}x = \int_a^b (y(t)\,x'(t) + x(t)\,y'(t))\,\mathrm{d}t \\ = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\,(x(t)\,y(t))\ \mathrm{d}t = x(t)\,y(t)|_a^b = 0,\end{aligned}$$

which then again yields $\int_\gamma \overline{z}\ \mathrm{d}z = 2i \cdot F(\gamma)$.