MA30056: Complex Analysis

EXERCISE SHEET 3: THE CAUCHY-RIEMANN EQUATIONS

Please hand solutions in at the lecture on Monday 16th February.

1.) Proof the *necessary Cauchy-Riemann condtion* (i.e., Theorem II.3.1).
Hint: Compute $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ twice, for $h = t \in \mathbb{R}$ and for $h = it$ purely imaginary.

2.) Show that (the continuous extension of) $f(z) = \frac{z^5}{|z|^4}$ is not (complex) differentiable at $z = 0$ but satisfies the Cauchy-Riemann equations there. Can you find similar functions with this property?

3.) Instead of writing a mapping in terms of its real and imaginary parts (i.e., $f = u + iv$), it is sometimes more convenient to write it in terms of modulus and argument:

$$f(z) = f(x+iy) = R(x,y)\ (\cos\Psi(x,y) + i\ \sin\Psi(x,y)) = R(x,y)\ \mathrm{e}^{i\Psi(x,y)}\ .$$

Show that in this case the Cauchy-Riemann equations read

$$\partial_x R = R \cdot \partial_y \Psi \qquad \text{and} \qquad \partial_y R = -R \cdot \partial_x \Psi.$$

4.) Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic. Prove that f is constant as soon as any one of $\mathrm{Re}\, f$, $\mathrm{Im}\, f$ or $|f|$ is constant.

5.) Prove Corollary II.3.4 under the additional assumption that the second partial derivatives of $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ exist and are continuous.

Please turn over!

Optional questions:

6.) (i) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show: f is differentiable everywhere but f' is not continuous.

(ii) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2 \sin(1/z)$ for $z \neq 0$, $f(0) = 0$. The solution to part (i) almost seems to give a proof that f is (complex) differentiable everywhere but f' is not continuous at the origin. This is impossible – where does the "proof" fail?

7.) Let $f(x+iy) = u(x,y) + i\,v(x,y)$ be holomorphic in a domain D, and let $(x_0, y_0) \in D$ where the gradient vectors of u and v do not vanish. Set $u_0 = u(x_0, y_0)$ and $v_0 = v(x_0, y_0)$. Show that the *level curves* $u(x,y) = u_0$ and $v(x,y) = v_0$ intersect perpendicularly at (x_0, y_0).

Show a similar result if one uses $f(x+iy) = R(x,y)\ \mathrm{e}^{i\Psi(x,y)}$.

Hint: Use the *Implicit Function Theorem on* $\mathbb{R}^2$: Let $g : \mathbb{R}^2 \supset [a,b] \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto g(x,y)$ be a continous function with $0 < m \leq \frac{\partial g}{\partial x} \leq M$ for some constants m, M, and all $x \in [a,b]$ and $y \in \mathbb{R}$. Then there exists a unique continuous function $h : [a,b] \to \mathbb{R}$ s.t. $g(x, h(x)) = 0$ for all $x \in [a,b]$, i.e., the equation $g(x,y) = 0$ implicitly defines a unique continuous function $h(x)$.