MA30056: Complex Analysis

Exercise Sheet 3: The Cauchy-Riemann equations

Please hand solutions in at the lecture on Monday 16th February.

1.) Proof the *necessary Cauchy-Riemann condtion* (i.e., Theorem II.3.1).
Hint: Compute $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ twice, for $h = t \in \mathbb{R}$ and for $h = it$ purely imaginary.

Solution:

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \\
&= \lim_{h\to 0} \frac{u(z+h)-u(z)+i(v(z+h)-v(z))}{h} \\
&= \begin{cases} \lim_{t\to 0}\left(\frac{u(x+t,y)-u(x,y)}{t} + i\frac{v(x+t,y)-v(x,y)}{t}\right), & h = t \in \mathbb{R} \\ \lim_{t\to 0}\left\{\frac{u(x,y+t)-u(x,y)}{it} + \frac{v(x,y+t)-v(x,y)}{t}\right\}, & h = it \in i\mathbb{R} \end{cases} \\
&= \begin{cases} \lim_{t\to 0}\frac{u(x+t,y)-u(x,y)}{t} + i\lim_{t\to 0}\frac{v(x+t,y)-v(x,y)}{t}, & h = t \in \mathbb{R} \\ -i\lim_{t\to 0}\frac{u(x,y+t)-u(x,y)}{t} + \lim_{t\to 0}\frac{v(x,y+t)-v(x,y)}{t}, & h = it \in i\mathbb{R} \end{cases} \\
&= \begin{cases} u_x + iv_x, & h = t \in \mathbb{R} \\ -iu_y + v_y, & h = it \in i\mathbb{R}. \end{cases}
\end{aligned}$$

Comparing real and imaginary parts in these equations yields the Cauchy-Riemann equations.

Note that u_x, v_x, u_y, v_y exist since f' exists (f is holomorphic).

2.) Show that (the continuous extension of) $f(z) = \frac{z^5}{|z|^4}$ is not (complex) differentiable at $z = 0$ but satisfies the Cauchy-Riemann equations there. Can you find similar functions with this property?

Solution: First consider

$$\lim_{r\to 0} \frac{f(r\,\mathrm{e}^{i\vartheta}) - 0}{r\,\mathrm{e}^{i\vartheta}} = \lim_{r\to 0} \mathrm{e}^{4i\vartheta}, \tag{1}$$

showing that f (or rather the continuous extension of f where $f(0) = 0$) is not differentiable at $z = 0$ since the limit depends on the direction $\mathrm{e}^{i\vartheta}$ and hence the limit $\lim_{h\to 0} \frac{f(h)-0}{h}$ with complex h does not exist.

Please turn over!

On the other hand, the above limit becomes 1 for $\vartheta = 0$ and $\vartheta = \frac{\pi}{2}$ (and also for $\vartheta = \pi, \frac{3\pi}{2}$) showing that[1]

$$u_x + iv_x = f_x = 1 \qquad \text{and} \qquad u_y + iv_y = f_y = i \lim_{y\to 0} \frac{f(iy) - f(0)}{iy} = i$$

(and thus $u_x = 1$, $v_x = 0$, $u_y = 0$ and v_y=1) so that f satisfies the Cauchy-Riemann equations at 0.

Further examples: $f(z) = z^{4k+1}/|z|^{4k}$, where $k \in \mathbb{N}$ (which in Equation (1) yields $\lim_{r\to 0} \mathrm{e}^{4k\, i\vartheta}$).

3.) Instead of writing a mapping in terms of its real and imaginary parts (i.e., $f = u + iv$), it is sometimes more convenient to write it in terms of modulus and argument:

$$f(z) = f(x+iy) = R(x,y)\ (\cos\Psi(x,y) + i\,\sin\Psi(x,y)) = R(x,y)\ \mathrm{e}^{i\Psi(x,y)}\,.$$

Show that in this case the Cauchy-Riemann equations read

$$\partial_x R = R \cdot \partial_y \Psi \qquad \text{and} \qquad \partial_y R = -R \cdot \partial_x \Psi.$$

Solution: We again[2] compute $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ twice, for $h = t \in \mathbb{R}$ and for $h = it$ purely imaginary.

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{R(z+h)\ \mathrm{e}^{i\Psi(z+h)} - R(z)\ \mathrm{e}^{i\Psi(z)}}{h} \\ &= \begin{cases} \lim_{t\to 0} \frac{R(x+t,y)\ \mathrm{e}^{i\Psi(x+t,y)} - R(x,y)\ \mathrm{e}^{i\Psi(x,y)}}{t} & \text{if } h = t \in \mathbb{R}, \\ \lim_{t\to 0} \frac{R(x,y+t)\ \mathrm{e}^{i\Psi(x,y+t)} - R(x,y)\ \mathrm{e}^{i\Psi(x,y)}}{it} & \text{if } h = it \in i\mathbb{R}, \end{cases} \end{aligned}$$

Please turn over!

[1] If you are confused by the "i", observe that in Question 1 we have shown that

$$f'(z) = \begin{cases} \partial_x f & \text{if } h = t \in \mathbb{R}, \\ -i\,\partial_y f & \text{if } h = i\,t \in i\mathbb{R}. \end{cases}$$

[2] Alternatively: Setting $u = R\,\cos\Psi$ and $v = R\,\sin\Psi$, the usual Cauchy-Riemann equations yield

$$\begin{aligned} u_x &= R_x \cdot \cos\Psi - R \cdot \Psi_x \cdot \sin\Psi &=& \quad R_y \cdot \sin\Psi + R \cdot \Psi_y \cdot \cos\Psi = v_y \\ u_y &= R_y \cdot \cos\Psi - R \cdot \Psi_y \cdot \sin\Psi &=& \quad -R_x \cdot \sin\Psi - R \cdot \Psi_x \cdot \cos\Psi = -v_x. \end{aligned}$$

Adding $cos\Psi$ times the first equation to $(-\sin\Psi)$ times the second equation and adding $\sin\Psi$ times the first equation to $\cos\Psi$ times the second equation yields the stated Cauchy-Riemann equations for R and Ψ.

$$= \begin{cases} \partial_x(R\, \mathrm{e}^{i\Psi}) & \text{if } h = t \in \mathbb{R}, \\ -i\, \partial_y(R\, \mathrm{e}^{i\Psi}) & \text{if } h = it \in i\mathbb{R}, \end{cases}$$

$$= \begin{cases} R_x\, \mathrm{e}^{i\Psi} + iR\Psi_x\, \mathrm{e}^{i\Psi} & \text{if } h = t \in \mathbb{R}, \\ -i\, R_y\, \mathrm{e}^{i\Psi} + R\Psi_y\, \mathrm{e}^{i\Psi} & \text{if } h = it \in i\mathbb{R}. \end{cases}$$

If f is differentiable, both results must be the same, so we get

$$R_x\, \mathrm{e}^{i\Psi} + iR\Psi_x\, \mathrm{e}^{i\Psi} = -i\, R_y\, \mathrm{e}^{i\Psi} + R\Psi_y\, \mathrm{e}^{i\Psi},$$

and therefore (since $\mathrm{e}^{i\Psi} \neq 0$, note that $|\, \mathrm{e}^{i\Psi}\,| = 1$)

$$R_x + iR\Psi_x = -i\, R_y + R\Psi_y.$$

Comparing real and imaginary parts in this equation yields the stated Cauchy-Riemann equations.

4.) Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic. Prove that f is constant as soon as any one of $\operatorname{Re} f$, $\operatorname{Im} f$ or $|f|$ is constant.

Solution: Write $f = u + iv$.

If $u = \operatorname{Re} f \equiv c$ then

$$f' = u_x + i\, v_x = u_x - i\, u_y = 0.$$

If $v = \operatorname{Im} f \equiv c$ then

$$f' = u_x + i\, v_x = v_y + i\, v_x = 0.$$

If $u^2 + v^2 = |f|^2 \equiv c^2$ then[3]

$$\left. \begin{array}{l} 0 = \frac{1}{2}|f|^2_x = uu_x + vv_x = uu_x - vu_y \\ 0 = \frac{1}{2}|f|^2_y = uu_y + vv_y = uu_y + vu_x \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{l} c^2 u_x = \quad \frac{1}{2}(u|f|^2_x + v|f|^2_y) = 0 \\ c^2 u_y = \frac{1}{2}(-v|f|^2_x + u|f|^2_y) = 0 \end{array} \right. .$$

Thus, if $\operatorname{Re} f \equiv c$, $\operatorname{Im} f \equiv c$ or $|f| \equiv c \neq 0$ then $f' \equiv 0$ and hence f is constant.

If, on the other hand, $|f| \equiv 0$ then $f \equiv 0$ is constant also.

5.) Prove Corollary II.3.4 under the additional assumption that the second partial derivatives of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ exist and are continuous.

Please turn over!

[3] Alternative: Using $f = R\, \mathrm{e}^{i\Psi}$, we have const. $= |f| = R$, thus $R_x = 0 = R_y$. With the Cauchy-Riemann equations $R_x = R\Psi_y$ and $R\Psi_x = -R_y$, it also follows $\Psi_x = 0 = \Psi_y$ if $R \neq 0$, and thus again $f' \equiv 0$.

Solution: Firstly,

$$\nabla v = (v_x, v_y) = (-u_y, u_x) = i \cdot (u_x, u_y) = i\,\nabla u$$

by the (necessary) Cauchy-Riemann conditions. Thus, if u and v are harmonic, then v is a conjugate harmonic function of u.

If the second partial derivatives of u and v exist and are continuous then Clairaut's theorem/Schwarz' lemma holds, i.e., the second mixed partial derivative commute.

Hence

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0 \qquad \text{and} \\ \Delta v &= v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = 0 \end{aligned}$$

by the (necessary) Cauchy-Riemann conditions and by Clairaut's theorem/Schwarz' lemma.

Here we have to know that f' is also holomorphic! Another possibility is to write $f'' = (f')'$ in different ways, using the necessary and sufficient Cauchy-Riemann conditions:

a) the second partial derivatives of u and v are continuous and hence $(u_x)_x = (v_y)_x = (v_x)_y$ and $(u_x)_y = (u_y)_x = -(v_x)_x$ so that $f' = u_x + iv_x$ is holomorphic by the sufficient Cauchy-Riemann conditions; then
b) from $f' = u_x + iv_x$ we get $f'' = u_{xx} + iv_{xx}$, by the necessary Cauchy-Riemann equations, and
c) from $f' = v_y - iu_y$ we get $f'' = -u_{yy} - iv_{yy}$, by the necessary Cauchy-Riemann equations again.

Subtracting we get $0 = f'' - f'' = \Delta u + i\Delta v$.

Optional questions:

6.) (i) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show: f is differentiable everywhere but f' is not continuous.

(ii) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2 \sin(1/z)$ for $z \neq 0$, $f(0) = 0$. The solution to part (i) almost seems to give a proof that f is (complex) differentiable everywhere but f' is not continuous at the origin. This is impossible – where does the "proof" fail?

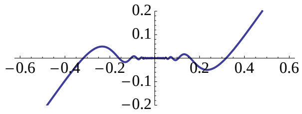
Please turn over!

Solution:

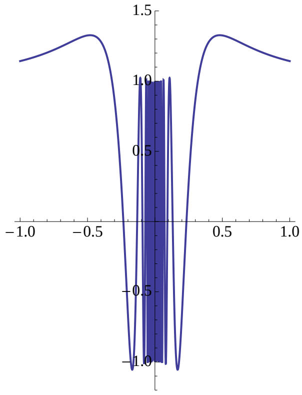
(i) If $x \neq 0$ then one simply calculates that $f'(x) = 2x\,\sin(1/x) - \cos(1/x)$; hence $\lim_{x\to 0} f'(x)$ does not exist (since $\lim_{x\to 0}\cos(1/x)$ does not exist, compare this with the function $x \to \sin(1/x)$ in Question 7 on Exercise sheet 2; depending on the sequence approaching zero you chose,$\cos(1/x)$ might converging to any value in $[-1,1]$ or not converge at all!), so f' is certainly not continuous at the origin. However, f *is*(!) differentiable at the origin. In fact, if $h \neq 0$ then

$$\left|\frac{f(h)-f(0)}{h-0}\right| = \left|\frac{h^2\,\sin(1/h)}{h}\right| \leq |h|, \qquad (2)$$

which shows that $f'(0) = 0$.



The graph of the function $f(x) = x^2\,\sin(1/x)$.



The graph of the derivative $f'(x)$. Surprisingly, we have $f'(0) = 0$.

(ii) We check whether $f(z)$ is (complex) differentiable at the origin. We consider the limit $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$: Eq. (2) establishes that for real h, this limit yields 0. If $h = i\,t$ ($t \in \mathbb{R}$) is imaginary, then we get

$$\lim_{h=it\to 0}\frac{f(h)-f(0)}{h} = \lim_{t\to 0}\frac{f(it)-f(0)}{it} = \lim_{t\to 0}\frac{-t^2\,\sin(-i/t)}{it}$$
$$= \lim_{t\to 0} it\,(-i\sinh(1/t)) = \lim_{t\to 0} t\,\sinh(1/t) = +\infty$$

For the limit observe that $\lim_{t\to 0^+} t\,\sinh(1/t) = \lim_{x\to+\infty}\sinh(x)/x = \lim_{x\to+\infty}(\mathrm{e}^x - \mathrm{e}^{-x})/(2\,x)$ and $\lim_{t\to 0^-} t\,\sinh(1/t) = \lim_{x\to-\infty}(\mathrm{e}^x - \mathrm{e}^{-x})/(2\,x) = \lim_{x\to+\infty}(\mathrm{e}^x - \mathrm{e}^{-x})/(2\,x)$. Then observe that e^x grows faster than any power of x.

(The point here is that the complex sine $\sin z$ is unbounded if $|\,\mathrm{Im}\,z| \to \infty$.)

Remark: That the (real) function $f(x) = x\,\sin(1/x)$ is not differentiable at 0 while $g(x) = x^2\,\sin(1/x)$ is, is demonstrated geometrically in “mathlets” (“mathematical applets”) that can be found at (in the section “Single Variable Calculus”)

http://www.math.umn.edu/~rogness/mathlets.shtml#single

7.) Let $f(x+iy) = u(x,y) + i\,v(x,y)$ be holomorphic in a domain D, and let $(x_0, y_0) \in D$ where the gradient vectors of u and v do not vanish. Set $u_0 = u(x_0, y_0)$ and $v_0 = v(x_0, y_0)$. Show that the *level curves* $u(x,y) = u_0$ and $v(x,y) = v_0$ intersect perpendicularly at (x_0, y_0).

Please turn over!

Show a similar result if one uses $f(x+iy) = R(x,y)\ \mathrm{e}^{i\Psi(x,y)}$.

Hint: Use the *Implicit Function Theorem on* $\mathbb{R}^2$: Let $g : \mathbb{R}^2 \supset [a,b] \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto g(x,y)$ be a continous function with $0 < m \leq \frac{\partial g}{\partial x} \leq M$ for some constants m, M, and all $x \in [a,b]$ and $y \in \mathbb{R}$. Then there exists a unique continuous function $h : [a,b] \to \mathbb{R}$ s.t. $g(x, h(x)) = 0$ for all $x \in [a,b]$, i.e., the equation $g(x,y) = 0$ implicitly defines a unique continuous function $h(x)$.

Solution:

- We first note that the notation $u(x,y) = u_0$ is sloppy for $\{(x,y) \mid u(x,y) = u_0\}$, i.e., for $\{z \in \mathbb{C} \mid \operatorname{Re} f(z) = u_0\}$. Similarly, $v(x,y) = v_0$ is sloppy for $\{z \in \mathbb{C} \mid \operatorname{Im} f(z) = v_0\}$.
- We use the following fact (a proof can be found below[4]): The gradient of a function $u(x,y)$ at (x_0, y_0) is orthogonal to the level curve of u at (x_0, y_0). Consequently, to show that the level curves of u and v meet orthogonally, it suffices to show that their gradient vectors at (x_0, y_0) are orthogonal.
- The gradient vectors of u and v are perpendicular since their scalar product vanishes:

$$\nabla u \cdot \nabla v = u_x\, v_x + u_y\, v_y \overset{\text{C-R}}{=} u_x\, (-u_y) + u_y\, (u_x) = 0.$$

  Similarly, using the Cauchy-Riemann equations $R_x = R\Psi_y$ and $R_y = -R\Psi_x$ (with the assumption that the gradients do not vanish and $R(x_0, y_0) \neq 0$), we get

$$\nabla R \cdot \nabla \Psi = R_x\, \Psi_x + R_y\, \Psi_y \overset{\text{C-R}}{=} R\Psi_y\, \Psi_x - R\Psi_x\, \Psi_y = 0.$$

  So, the level curves $\{z \in \mathbb{C} \mid |f(z)| = R_0\}$ and $\{z \in \mathbb{C} \mid \arg f(z) = \Psi_0\}$ are perpendicular to each other.
- For pictures of level curves: see lecture notes, Sections II.3 & II.4.

[4] This is an easy applicaton of the *implicit function theorem* which we discus informally in the following: Locally around a point (x_0, y_0) the stated implicit function theorem holds (we might have to consider $(-f)$), since either $\partial_x f \neq 0$ or $\partial_y f \neq 0$ in some neighbourhood of (x_0, y_0) (by continuity of these derivatives and since the gradient at (x_0, y_0) does not vanish). Thus, we can describe a level set $u(x,y) = u(x_0, y_0)$ by a function $y = h(x)$ (or a function $x = \tilde{h}(y)$). For such a function we have

$$u(x, h(x)) - u(x_0, y_0) = 0$$

(and, of course, $y_0 = h(x_0)$). Differentiation of this equation by x yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} \left(u(x, h(x)) - u(x_0, y_0)\right)\Big|_{x = x_0} = \partial_x f(x_0, y_0) + \partial_y f(x_0, y_0)\, h'(x_0).$$

If $\partial_y f(x_0, y_0) \neq 0$, this gives $h'(x_0)$ (otherwise, note that by assumption $\partial_x f(x_0, y_0) \neq 0$ since the gradient does not vanish, and use $\tilde{h}$). So, in the neighbourhood of (x_0, y_0) the map $x \mapsto (x, h(x))$ describes a level set. Since its tangent at x_0 is $(1, h'(x_0))$, the previous equation shows that the gradient is orthogonal to the (tangent on the) level set.

Remark (for students who have done M41): The implicit function theorem can be proved using *Banach's Contraction Mapping Principle* which also gives us an iterative method to calculate these level sets.