MA30056: Complex Analysis

Exercise Sheet 2: Paths, complex functions and complex differentiability

Please hand solutions in at the lecture on Monday 9th February.

1.) Prove that the composition of two (piecewise smooth) paths is a (piecewise smooth) path.

2.) Show that a *Möbius transformation* $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$, maps lines and circles to lines and circles.
Hint: $\alpha|z|^2 + \beta(z + \overline{z}) + i\gamma(z - \overline{z}) + \delta = 0$ is the equation of a circle or a line in $\mathbb{C}$; investigate $f(z) = \frac{1}{z}$ and write $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c}\frac{1}{cz+d}$.

3.) Show that

$$\mathbb{C} \setminus \{0\} \ni z = x + iy \mapsto f(z) = \frac{x^2 y}{x^4 + y^2} \in \mathbb{R} \subset \mathbb{C}$$

has no continuous extension to the whole plane.

4.) Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be continuous and $K \subset D$ compact. Show that $f(K) \subset \mathbb{C}$ is compact and, in particular, bounded.

5.) Show that $\mathbb{C} \ni z \mapsto f(z) = |z|^2$ is differentiable at $z = 0$ only.

Optional question:

6.) Prove Theorem I.5.1 & Lemma I.5.2.

Very optional question:

7.) Show, using the example

$$A = \left\{ z = x + iy \in \mathbb{C} \mid (x = 0 \text{ and } y \in [-1, 1]) \vee \left(y = \sin \frac{1}{x} \right) \right\},$$

that connectedness does not imply path connectedness, i.e., show that A is connected but not path connected.