MA30056: Complex Analysis

Exercise Sheet 2: Paths, complex functions and complex differentiability

Please hand solutions in at the lecture on Monday 9th February.

1.) Prove that the composition of two (piecewise smooth) paths is a (piecewise smooth) path.

Solution: There are two issues: the composition is a path (i.e., continuous) and, if both paths were piecewise smooth, then the composition is piecewise smooth (i.e., composition of finitely many smooth paths).

- *The composition of two paths is a path:* Take $\gamma = \gamma_1 + \gamma_2$, where $\gamma_i : [a_i, b_i] \to \mathbb{C}$ are paths.
  Fix $t \in [a_1, b_1 + (b_2 - a_2)]$.
  If $t \neq b_1$ then

$$\lim_{s \to t} \gamma(s) = \begin{cases} \lim_{s\to t} \gamma_1(s) = \gamma_1(t) = \gamma(t) & \text{if } t < b_1, \\ \lim_{s\to t} \gamma_2(s - b_1 + a_2) = \gamma_2(t - b_1 + a_2) = \gamma(t) & \text{if } t > b_1, \end{cases}$$

  since γ_1 and γ_2 (and $t \mapsto t - b_1 + a_2$) are continuous.
  If $t = b_1$ then

$$\left.\begin{array}{l} \lim_{s\nearrow t} \gamma(s) = \lim_{s\nearrow t} \gamma_1(s) = \gamma_1(b_1) \\ \lim_{s\searrow t} \gamma(s) = \lim_{s\searrow t} \gamma_2(s - b_1 + a_2) = \gamma_2(a_2) \end{array}\right\} \quad \Rightarrow \quad \lim_{s\to t} \gamma(s) = \gamma(t)$$

  since the γ_i are continuous and $\gamma_1(b_1) = \gamma_2(a_2)$. (Note that this is where we need $\gamma_1(b_1) = \gamma_2(a_2)$!)

- *If γ_1 and γ_2 are piecewise smooth then γ is piecewise smooth:* For this, first note that path composition is associative: $(\gamma_1 + \gamma_2) + \gamma_3 = \gamma_1 + (\gamma_2 + \gamma_3)$ – this is why it makes sense to write $\gamma_1 + \gamma_2 + \gamma_3$ in the first place.
  Now, if $\gamma_1 = \gamma_{1,1} + \cdots + \gamma_{1,m}$ and $\gamma_2 = \gamma_{2,1} + \cdots + \gamma_{2,n}$ are compositions of finitely many (m and $n \in \mathbb{N}$, respectively) smooth paths then

$$\gamma = \gamma_1 + \gamma_2 = \gamma_{1,1} + \cdots + \gamma_{1,m} + \gamma_{2,1} + \cdots + \gamma_{2,n}$$

  is a composition of finitely many ($m + n \in \mathbb{N}$) smooth paths.

2.) Show that a *Möbius transformation* $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$, maps lines and circles to lines and circles.
Hint: $\alpha|z|^2 + \beta(z + \overline{z}) + i\gamma(z - \overline{z}) + \delta = 0$ is the equation of a circle or a line in $\mathbb{C}$; investigate $f(z) = \frac{1}{z}$ and write $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c}\frac{1}{cz+d}$.

Solution: It is quite a computation to prove this by pure computation. Thus, to simplify life, we only do the computation for a special case and argue that this is enough.

If $c = 0$ then $z \mapsto \frac{a}{d}z + \frac{b}{d}$ is a stretch-rotation followed by a translation and, therefore, maps lines to lines and circles to circles.

If $c \neq 0$ we write $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c}\frac{1}{cz+d}$: thus, the Möbius transformation is a composition of a stretch-rotation followed by a translation, followed by the *inversion* $z \mapsto \frac{1}{z}$, followed by a stretch-rotation and a translation again. Hence it suffices to prove the claim for the inversion $z \mapsto \frac{1}{z}$.

For this, write $w = \frac{1}{z}$ and compute

$$A|w|^2 + B(w + \overline{w}) + iC(w - \overline{w}) + D = \frac{1}{|z|^2}\{D|z|^2 + B(z + \overline{z}) - iC(z - \overline{z}) + A\} = 0$$

with $D = \alpha$, $B = \beta$, $C = -\gamma$ and $A = \delta$ when z satisfies the circle equation in the hint.

Remark: Note that we have $a, b, c, d \in \mathbb{C}$, while $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Furthermore, The circle equation actually can be rewritten as $\alpha x^2 + \alpha y^2 + 2\beta x - 2\gamma y + \delta = 0$ and one recognises the familiar equation for circles or (if $\alpha = 0$) lines in the plane.

Note: Möbius transformation form a group, in fact, the group of Möbius transformations is exactly the automorphism group of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (meaning that the Möbius transformations are characterised as invertible holomorphic functions $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with holomorphic inverse – note however that a function that is holomorphic as a function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is not necessarily holomorphic as a function $\mathbb{C} \to \mathbb{C}$, e.g., $f(z) = \frac{az+b}{cz+d}$ is (usually) not holomorphic at $z = -\frac{d}{c}$). This connects topology (of the sphere) with algebra (groups) via complex analysis!

Moreover, besides the identity map, a Möbius transformation has at most two distinct fixed points in $\hat{\mathbb{C}}$; this has the consequence that two Möbius transformations are equal if the are equal at three(!) distinct points.

More on Möbius transformations (and also its connection to non-Euclidean geometries) can be found in Chapter 3 of T. Needham: Visual Complex Analysis; Clarendon Press, Oxford (1997); library: `513.317 NEE`.

A really nice video about Möbius transformations is available at

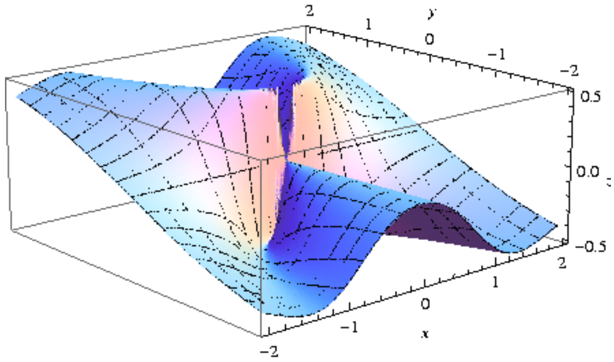
http://www.youtube.com/watch?v=JX3VmDgiFnY&NR=1

3.) Show that

$$\mathbb{C} \setminus \{0\} \ni z = x + iy \mapsto f(z) = \frac{x^2 y}{x^4 + y^2} \in \mathbb{R} \subset \mathbb{C}$$

has no continuous extension to the whole plane.

Solution: We compare the limit we get when approaching 0 on different paths: If $\gamma : t \mapsto t$ and $\tilde{\gamma} : t \mapsto it$, then we get $\lim_{t\to 0} f(\gamma(t)) = 0 = \lim_{t\to 0} f(\tilde{\gamma}(t))$. However, if we consider the path $\hat{\gamma} : t \mapsto t + it^2$ then we have $\lim_{t\to 0} f(\hat{\gamma}(t)) = \frac{1}{2}$. Thus, f cannot be continuous at 0.
The graph of f is shown on the right.



4.) Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be continuous and $K \subset D$ compact. Show that $f(K) \subset \mathbb{C}$ is compact and, in particular, bounded.

Solution: There are many valid solutions for this problem of varying sophistication.

- *A proof that requires only M7/M11 technology.* Observe: If $(w_n)_{n\in\mathbb{N}}$ is a sequence in $f(K)$ then there is a sequence $(z_n)_{n\in\mathbb{N}} \subset K$ with $f(z_n) = w_n$ for all $n \in \mathbb{N}$ (note that this sequence will usually not be unique). Now, K is compact, that is,

  a) K is bounded – therefore there is a convergent subsequence $(z_{n_k})_{k\in\mathbb{N}}$, $z_{n_k} \to z \in \mathbb{C}$, by Bolzano-Weierstrass' theorem, and
  b) K is closed – hence $z \in K$.

  Since f is continuous $w_{n_k} = f(z_{n_k}) \to w = f(z) \in f(K)$.
  $f(K)$ is closed: Now take $(w_n)_{n\in\mathbb{N}}$ convergent to some $w \in \mathbb{C}$. By the above observation some subsequence $(w_{n_k})_{k\in\mathbb{N}}$ converges to $f(z)$ – but any subsequence of $(w_n)_{n\in\mathbb{N}}$ converges to w. Hence $w = f(z) \in f(K)$.
  $f(K)$ is bounded: Suppose not. Then there would be a sequence $(w_n)_{n\in\mathbb{N}} \subset f(K)$ with $|w_n| \geq n$ for all $n \in \mathbb{N}$. However, this sequence cannot contain a convergent subsequence as any convergent sequence is bounded – this contradicts the above observation.

Please turn over!

- *A proof using topological techniques.* First recall:
  a) a set is compact if every open covering has a finite subcovering; and
  b) a map is continuous if the pre-image of any open set is open.

  Now take an open covering $(U_i)_{i\in I}$, I some index set (doesn't have to be countable), of $f(K)$.
  By continuity of f, $(f^{-1}(U_i))_{i\in I}$ is an open covering of K which, by compactness of K, has a finite subcovering $(f^{-1}(U_{i_k}))_{k=1,\ldots,n}$.
  Thus $(U_{i_k})_{k=1,\ldots,n}$ is a finite subcovering (if $w = f(z) \in f(K)$ then there is a k so that $z \in f^{-1}(U_{i_k})$ since the $f^{-1}(U_{i_k})$ cover K – therefore $w \in U_{i_k}$ as $f^{-1}(U_{i_k}) = \{z \mid f(z) \in U_{i_k}\}$).

5.) Show that $\mathbb{C} \ni z \mapsto f(z) = |z|^2$ is differentiable at $z = 0$ only.

Solution: We go back to the definition of complex differentiability: consider

$$\frac{|z+h|^2 - |z|^2}{h} = \frac{z\overline{h} + \overline{z}h + h\overline{h}}{h} = z\frac{\overline{h}}{h} + \overline{z} + \overline{h} = \eta_{z,h}.$$

Now, if $z = 0$, then $\eta_{0,h} = \overline{h} \to 0$ as $h \to 0$ ($h \mapsto \overline{h}$ is continuous) and, hence, f is differentiable at $z = 0$ with $f'(0) = 0$.

If, on the other hand, $z \neq 0$ then

$$\lim_{h\to 0, h\in\mathbb{R}} \eta_{z,h} = z + \overline{z} \qquad \neq \qquad -z + \overline{z} = \lim_{h\to 0, h\in i\mathbb{R}} \eta_{z,h}$$

and, hence, f is not differentiable at $z \neq 0$.

Optional question:

6.) Prove Theorem I.5.1 & Lemma I.5.2.

Solution: *Proof of Theorem I.5.1:*
Let A be connected and let $z_0 \in A$ and define the *connected component* of the point z_0 by $A_1 = \{z \in A \mid \exists \text{ a path joining } z \text{ and } z_0\}$ and set $A_2 = A \setminus A_1$. Then, one can show that A_1, A_2 are open: Let $z \in A_1$. Then there exists a path γ in A_1 from z_0 to z. Since A is open, there exists $\varepsilon > 0$ s.t. $B_\varepsilon(z) \subset A$. Let $\tilde{z} \in B_\varepsilon(z)$. Then there is a path in $B_\varepsilon(z)$ from z to $\tilde{z}$, namely $\tilde{\gamma}(t) = (1-t)\,z + t\,\tilde{z}$ (the straight line joining them). But $\gamma + \tilde{\gamma}$ is a path in A from z_0 to $\tilde{z}$. Hence $\tilde{z} \in A_1$, so A_1 is open. Now A_2 is the union of the open (as we have just seen for A_1) connected components besides A_1 and thus open (alternatively, take any $z \in A_2$ and let $\varepsilon > 0$ be such that $B_\varepsilon(z) \subset A$; then one can show that in fact $B_\varepsilon(z) \subset A_2$).

Please turn over!

Therefore A_1, A_2 are open and closed in(!) A. They are also disjoint and their union is all of A. So, since A is connected and A_1 is nonempty, A_2 must be the empty set. Thus, $A = A_1$ and A is path connected.

One possibility for the converse is the following:
Suppose A is pathwise connected and let $z_0 \in A$. Then, for each $z \in A$, we find a path γ_z in A joining z and z_0 and the corresponding Jordan curve $\mathit{\Gamma}_z$ is a connected set (as continuous image of an interval which, after all, is connected). Thus we have $\bigcup_{z\in A} \mathit{\Gamma}_z = A$ and $z_0 \in \bigcap_{z\in A} \mathit{\Gamma}_z$. One can then show (or use): the union of arbitrarily many connected sets with nonempty intersection is connected.

Proof of Lemma I.5.2:
Assume that A is path-connected, and the closed and open subset A_1 is neither empty nor all of A. Then its complement $A_2 = A \setminus A_1$ is also nonempty, and we can choose $z \in A_1$ and $w \in A_2$ and, furthermore (by path-connectedness of A) a path γ connecting z and w, say with $\gamma(0) = z$ and $\gamma(1) = w$.

Now, we want to obtain a contradiction: For this, consider the set $T = \{t \in [0,1] \mid \gamma(t) \in A_1\}$ and set $r = \sup T$ (which exists by the completeness of the real numbers).

Since A_1 is open, $\gamma(r) \notin A_1$: Since A_1 is open, for every point in A_1 we find an open disk around that point contained entirely in A_1. So, by the continuity of γ, if $\gamma(s)$ is in A_1, also $\gamma(s+\delta)$ for some $\delta > 0$ is in A_1 (this shows that $\gamma(r)$ cannot be in A_1: $\gamma(r+\delta)$ is not in A_1 for any $0 < \delta \leq 1 - r$).

Since A_1 is closed, $\gamma(r) \in A_1$: If A_1 is closed, its complement A_2 is open. Applying the previous argument to A_2 (with the necessary changes) shows that $\gamma(r)$ is not in A_2, consequently $\gamma(r)$ must be in A_1.

Thus we have a contradiction, and A_1 must either be empty or all of A.

Very optional question:

7.) Show, using the example

$$A = \left\{ z = x + iy \in \mathbb{C} \mid (x = 0 \text{ and } y \in [-1,1]) \vee \left(y = \sin\frac{1}{x} \right) \right\},$$

that connectedness does not imply path connectedness, i.e., show that A is connected but not path connected.

Please turn over!

Solution: This needs some results from a unit like "Metric Spaces". We try to give some details here. We let $\tilde{A} = \{z = x + iy \in \mathbb{C} \mid y = \sin\frac{1}{x}, x > 0\}$, the graph of $\sin(1/x)$ for $x > 0$.

<u>A is connected:</u> We first note that $\tilde{A}$ is connected, since it is the continuous image of an interval (i.e., it is the continuous image of a connected set and thus connected), i.e., using the continuous(!) map $g : (0, \infty) \to \mathbb{C}$, $g(t) = t + i\sin(1/t)$, one has $\tilde{A} = g((0, \infty))$.

One can show that $\tilde{A} \cup \{iy \mid y \in [-1, 1]\}$ is the closure of $\tilde{A}$: For example, the sequence $((1/(\frac{\pi}{2} + 2\pi n), 1))$ (the sequence of maxima in the graph) converges to i. Similarly,

$$\tilde{A} \ni \frac{1}{\arcsin(y) + 2\pi n} + i\,y \;\to\; i\,y$$

However, any point outside the graph $\tilde{A}$ and outside $\{i\,y \mid -1 \leq y \leq 1\}$ is not a limit point.

A theorem in "Topology"/"Metric Spaces" states that if a set is connected so is its closure. Thus since $\tilde{A}$ is connected, so is $\tilde{A} \cup \{iy \mid y \in [-1, 1]\}$; and repeating the same argument with the left half ("$x < 0$"-part) of the graph and observing that the union of two connected sets with nonempty intersection (here, this is the set $\{iy \mid y \in [-1, 1]\}$) is again connected, yields that A is connected.

<u>A is not path connected:</u> we recall that a path has to be a continuous function, in this case a continuous function $\gamma : [0, 1] \to \mathbb{R}^2 \cong \mathbb{C}$. Now show that there is no *continuous* function with $\gamma(0) = 0 + i \cdot y$ and $\gamma(1) = s + i \cdot \sin\frac{1}{s}$ for some $s > 0$ and some $-1 \leq y \leq 1$:

Seeking a contradiction, we assume that it is path connected. Let $r = \sup\{t \in [0, 1] \mid \operatorname{Re}\gamma(t) = 0\}$ (i.e., the "path" γ changes from the connected set $\{iy \mid y \in [-1, 1]\}$ to the graph $\tilde{A}$ at r "for the last time"). Note that the set $\{t \in [0, 1] \mid \operatorname{Re}\gamma(t) = 0\}$ is closed (as preimage of a closed set), thus $r < 1$ and $\gamma(r) = 0$. Now, consider for some $\delta > 0$, the graph of $\gamma(t)$ for $t \in [r, r + \delta)$. Since $\operatorname{Re}\gamma(t) \neq 0$ for $t > r$, we have $\gamma(r + \frac{\delta}{2}) = 1/a + i\sin(a)$ for some $a > 0$. It is then clear that γ can only be continuous if all of $A' = \{x + i\sin(1/x) \mid x \in [1/(a + 2\pi), 1/a]\}$ is contained in the graph $\{\gamma(t) \mid t \in [r, r + \delta/2]\}$: Note that if γ is continuous, then the function that maps t to the real coordinate of $\gamma(t)$ is also continuous and we observe that the continuous image of a closed and bounded interval (i.e., compact and connected subset of $\mathbb{R}$) is a closed and bounded interval (here, the image of $[r, r + \delta/2]$ must contain $[0, 1/a]$). But U' contains "a period of sine", i.e., the second coordinate contains all values between -1 and 1. So, there is a $t_1 \in [r, r + \delta)$ s.t. $|\gamma(t_1) - \gamma(r)| \geq \frac{1}{2}$. Since this holds for any $\delta > 0$, the function γ is not continuous and thus contradicts the definition of a path, i.e., A is not path connected.

This is a bit technical. You can simplify it if you assume w.l.o.g. that $r = 0$ and $\gamma(0) = 0$.