MA30056: Complex Analysis

Exercise Sheet 1: Complex Numbers

Please hand solutions in at the lecture on Monday 2nd February.

1.) Let $z, w \in \mathbb{C}$. Verify that

   (i) $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$,

   (ii) $|z| = |\bar{z}|$, i.e., $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is an *isometry*,

   (iii) $|z \cdot w| = |z| \cdot |w|$, i.e., $|\,.\,| : (\mathbb{C}, \cdot) \to (\mathbb{R}, \cdot)$ is a *homomorphism*,

   (iv) $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.

   Conclude that the *triangle inequality* $|z + w| \leq |z| + |w|$ holds.
   Hint: Compute $|z + w|^2$.

2.) Show that the equation $z^n = 1$, $n \in \mathbb{N}$, has n solutions.
   Determine the solutions of $z^3 = 1$.

3.) Show that open/closed disks are open/closed.

4.) (*Cantor's Intersection Thm*)
   Let $K_n \subset \mathbb{C}$ be compact with $K_n \supset K_{n+1}$ for all n and

$$\operatorname{diam} K_n = \sup_{z,w \in K_n} |z - w| \to 0 \qquad \text{as } n \to \infty.$$

   Prove that $\exists z \in \mathbb{C} : \bigcap_{n \in \mathbb{N}} K_n = \{z\}$.

5.) Prove that $|\,.\,| : \mathbb{C} \to \mathbb{R}$ is continuous.

Optional question:

6.) Show that $\mathbb{C}$ is *not*(!) an *ordered field.*
   Note that an *ordering* of a field K is a subset $P \subset K$ having the following properties:

   (O1) Given $x \in K$, we have either $x \in P$, or $x = 0$ or $-x \in P$, and these three possibilities are mutually exclusive. In other words, K is the disjoint union of P, $\{0\}$ and $-P$.

   (O2) If $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$.

   We shall also say that K is *ordered by* P, and we call P the set of *positive elements.*
   Hint: Proof by contradiction.