## MA30056: Complex Analysis

Last Exercise Sheet: Residues

`Q: How does a mathematician call his dog?`

`A: Cauchy - because it leaves a residue at every pole...`

Solutions will be available from Friday 1st May.

1.) Each of the following functions $f$ has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that $f$ is holomorphic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity just state it.

(i) $f(z) = \frac{\cos(z)-1}{z}$

(ii) $f(z) = \mathrm{e}^{1/z}$

(iii) $f(z) = \frac{\cos(1/z)}{1/z}$

(iv) $f(z) = \frac{1}{1-\mathrm{e}^z}$

2.) Prove the Casorati-Weierstrass Theorem.
*Hint:* Proof by contradiction; fix $c$ and consider $g = \frac{1}{f-c}$.

3.) Prove the $p/q'$-rule: Suppose $p, q : B_R(z_0) \to \mathbb{C}$ are holomorphic and $q$ has a zero of order $n = 1$ at $z_0$, i.e., $q(z_0) = 0$ and $q'(z_0) \neq 0$. Then $f = \frac{p}{q}$ has $\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$.

4.) Evaluate $\int_0^\infty \frac{\cos x \, \mathrm{d}x}{(1+x^2)(4+x^2)}$ using the Theorem of Residues.
*Hint:* Consider $f(z) = \frac{\mathrm{e}^{iz}}{(z^2+1)(z^2+4)}$.

*Optional question:*

5.) Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.
*Hint:* Use the function $\frac{\pi}{z^2 \sin(\pi z)}$.