MA30056: Complex Analysis

Last Exercise Sheet: Residues

`Q: How does a mathematician call his dog?`

`A: Cauchy - because it leaves a residue at every pole...`

Solutions will be available from Friday 1st May.

1.) Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is holomorphic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity just state it.

(i) $f(z) = \frac{\cos(z)-1}{z}$

(ii) $f(z) = \mathrm{e}^{1/z}$

(iii) $f(z) = \frac{\cos(1/z)}{1/z}$

(iv) $f(z) = \frac{1}{1-\mathrm{e}^z}$

Solution:

(i) $\lim_{z\to 0} \frac{\cos(z)-1}{z} \overset{\text{de l'Hospital}}{=} \lim_{z\to 0} \frac{-\sin z}{1} = -\sin 0 = 0$. Therefore the singularity at $z = 0$ is removable and we define $f(0) = 0$.

(ii) $f(z) = \mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$, and so we have an infinite number of nonzero terms in the singular part. Therefore, the singularity at $z = 0$ is essential.

(iii) $f(z) = \frac{\cos(1/z)}{1/z} = z\,\cos(\frac{1}{z}) = z \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, z^{2n-1}}$ and so we have an infinite number of nonzero terms in the singular part. Therefore, the singularity at $z = 0$ is essential.

(iv) $|\frac{1}{1-\mathrm{e}^z}| \to \infty$ as $z \to 0$ and so there is a pole at $z = 0$.
Thus, $f(z) = \frac{1}{1-\mathrm{e}^z} = \frac{a_{-n}}{z^n} + \ldots + \frac{a_1}{z} + a_0 + a_1\,z + a_2\,z^2 + \ldots$. Now, for $n \geq 1$ we get $a_{-n} = \lim_{z\to 0} z^n\, f(z) = \lim_{z\to 0} \frac{z^n}{1-\mathrm{e}^z} \overset{\text{de l'Hospital}}{=} \lim_{z\to 0} \frac{n\,z^{n-1}}{-\mathrm{e}^z} = 0$ unless $n = 1$; in the case $n = 1$ we obtain $\lim_{z\to 0} z\, f(z) = \lim_{z\to 0} \frac{z}{1-\mathrm{e}^z} \overset{\text{de l'Hospital}}{=} \lim_{z\to 0} \frac{1}{-\mathrm{e}^z} = -1$. Thus, $f(z) = \frac{-1}{z} + a_0 + a_1\,z + a_2\,z^2 + \ldots$. Therefore, the pole at $z = 0$ is a simple pole and the singular part is $\frac{-1}{z}$.

2.) Prove the Casorati-Weierstrass Theorem.
Hint: Proof by contradiction; fix c and consider $g = \frac{1}{f-c}$.

Solution: Let z_0 be an essential singularity of f and suppose, for a contradiction, we can find a $c \in \mathbb{C}$ so that there are $\varepsilon > 0$ and $\delta > 0$ so that

$$|f(z) - c| \geq \varepsilon \qquad \text{for} \qquad 0 < |z - z_0| < \delta.$$

Thus the function

$$g : B_\delta^*(z_0), \quad z \mapsto g(z) = \frac{1}{f(z) - c}$$

is holomorphic in $B_\delta^*(z_0)$ and bounded, and therefore, it extends holomorphically to $B_\delta(z_0)$ by Theorem V.1.1 (i).

Now, if $g(z_0) \neq 0$ then

$$B_\delta(z_0) \ni z \mapsto f(z) = c + \frac{1}{g(z)} \in \mathbb{C}$$

is holomorphic, contradicting the assumption that f has an essential singularity at z_0.

If, on the other hand, $g(z_0) = 0$ then $\frac{1}{|g(z)|} \to \infty$ as $z \to z_0$ and therefore

$$B_\delta^*(z_0) \ni z \mapsto f(z) = c + \frac{1}{g(z)} \in \mathbb{C}$$

has, with $\frac{1}{g(z)}$, a pole at z_0 by Theorem V.1.1 (ii). Again, this contradicts the assumption that f has an *essential* singularity at z_0.

3.) Prove the p/q'-rule: Suppose $p, q : B_R(z_0) \to \mathbb{C}$ are holomorphic and q has a zero of order $n = 1$ at z_0, i.e., $q(z_0) = 0$ and $q'(z_0) \neq 0$. Then $f = \frac{p}{q}$ has $\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$.

Solution: To simplify notation we take, w.l.o.g., $z_0 = 0$. Since q has a zero of order $n = 1$ at $z = z_0 = 0$ we have $q(z) = \sum_{k=1}^\infty a_k z^k = z\, g(z)$ with $g(0) \neq 0$. Note that $q'(0) = a_1 = g(0)$. Hence $\frac{p}{g}$ is holomorphic in $B_R(z_0)$ so that, by the Cauchy-Taylor Theorem,

$$\frac{p(z)}{g(z)} = \sum_{k=0}^\infty b_k z^k \qquad \Rightarrow \qquad \frac{p(z)}{q(z)} = \frac{1}{z}\frac{p(z)}{g(z)} = \sum_{k=-1}^\infty b_{k+1} z^k$$

for $z \in B_R^*(0)$. Hence $\mathrm{Res}(\frac{p}{q}, 0) = b_0 = \lim_{z \to 0} \frac{p(z)}{g(z)} = \frac{p(0)}{g(0)} = \frac{p(0)}{q'(0)}$.

4.) Evaluate $\int_0^\infty \frac{\cos x\,\mathrm{d}x}{(1+x^2)(4+x^2)}$ using the Theorem of Residues.
Hint: Consider $f(z) = \frac{\mathrm{e}^{iz}}{(z^2+1)(z^2+4)}$.

Solution: Consider $f(z) = \frac{\mathrm{e}^{iz}}{(1+z^2)(4+z^2)}$ and compute

$$\int_{[-R,R]\cup\Gamma(R)} f(z)\,\mathrm{d}z, \qquad \text{where} \qquad \Gamma(R) = \gamma([0,\pi]), \qquad [0,\pi] \ni t \mapsto \gamma(t) = R\,\mathrm{e}^{it},$$

is the half circle with radius R and center $z = 0$ in the upper half plane.

First we need the residues of $f(z)$: since f has simple poles at $\pm i$ and $\pm 2i$ we compute

$$\begin{aligned}
\mathrm{Res}(f,i) &= \lim_{h\to 0} hf(i+h) \\
&= \lim_{h\to 0} \frac{h\,\mathrm{e}^{i(i+h)}}{(i+h-i)(i+h+i)(4+(i+h)^2)} \\
&= \frac{1}{6i\ \mathrm{e}}, \\
\mathrm{Res}(f,2i) &= \lim_{h\to 0} hf(2i+h) \\
&= \lim_{h\to 0} \frac{h\,\mathrm{e}^{i(2i+h)}}{(1+(2i+h)^2)(2i+h-2i)(2i+h+2i)} \\
&= \frac{-1}{12i\ \mathrm{e}^2};
\end{aligned}$$

since we are integrating in the upper half plane we are not going to need the other two residues.

Now, writing $z = x + iy$, we have

$$|\,\mathrm{e}^{iz}\,| = \mathrm{e}^{-y} \qquad \Rightarrow \qquad |f(z)| = \frac{\mathrm{e}^{-y}}{|z^2+1|\,|z^2+4|} \le \frac{\mathrm{e}^{-y}}{(|z|^2-1)(|z|^2-4)}$$

by the reversed triangle inequality when $|z| > 2$. Hence, for $z \in \Gamma(R)$ with $R > 2$,

$$|f(z)| \le \frac{1}{(R^2-1)(R^2-4)}$$

since $y \ge 0$ so that $\mathrm{e}^{-y} \le 1$. Hence the ML-inequality gives

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \frac{1}{(R^2-1)(R^2-4)}\,R\pi = \frac{\pi}{R^3(1-\frac{1}{R^2})(1-\frac{4}{R^2})} \to 0 \qquad \text{as} \qquad R \to \infty.$$

Please turn over!

Thus, by the Residue Theorem

$$\begin{aligned}\int_0^R \frac{\cos x}{(1+x^2)(4+x^2)}\,\mathrm{d}x &= \frac{1}{2}\int_{-R}^{R} \frac{\cos x + i\sin x}{(1+x^2)(4+x^2)}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{[-R,R]} f(z)\,\mathrm{d}z \\ &= \pi i(\operatorname{Res}(f,i)+\operatorname{Res}(f,2i)) - \frac{1}{2}\int_{\Gamma(R)} f(z)\,\mathrm{d}z \\ &\to \pi\left(\frac{1}{6\,\mathrm{e}} - \frac{1}{12\,\mathrm{e}^2}\right)\end{aligned}$$

as $R \to \infty$, where the first equality holds since $x \mapsto \sin x$ is an odd function (so that the imaginary part of the integral vanishes) and $x \mapsto \cos x$ is an even function.

(In particular we have also proved that the initial integral exists.)

Optional question:

5.) Show that $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.
Hint: Use the function $\frac{\pi}{z^2\sin(\pi z)}$.

Solution: This is very similar to the calculation of $\zeta(2)$. Indeed, in this application of the Residue Theorem we use the same contours, the "boxes" $\square_N = \{z \in \mathbb{C} \mid \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} = N + \frac{1}{2}\}$. We begin by calculating the residues of $f(z) = \frac{\pi}{z^2\sin(\pi z)}$ inside $\square_N$:

(i) We have $\sin(\pi\, n) = 0$ for all $n \in \mathbb{Z}$ (these are the only zeros of sine!). Noting that the derivative $\sin'(\pi\, n) = \pi\,\cos(\pi\, n) = (-1)^n\pi$ for $n \in \mathbb{Z}$ and these are thus simple zeros, we can use the above p/q'-rule to determine the residues of $f(z) = \frac{\pi}{z^2\sin(\pi z)}$ for all nonzero integers $n \in \mathbb{Z}\setminus\{0\}$ (taking $p(z) = \pi/z^2$ and $q(z) = \sin(\pi\, z)$):

$$\operatorname{Res}(f,n) = \frac{\pi}{n^2}\cdot\frac{1}{(-1)^n\pi} = \frac{(-1)^n}{n^2}.$$

(ii) We now determine the Laurent series of $f(z)$ around $z_0 = 0$. By Question 5 on Exercise sheet 9 (or the previous considerations), $1/\sin(z)$ has a simple pole at 0 of residue 1. Consequently, $1/\sin(\pi\, z)$ also has a simple pole at 0 and we have

$$\operatorname{Res}\left(\tfrac{\pi}{\sin(\pi\, z)}, 0\right) = \lim_{z\to 0}\frac{\pi\, z}{\sin(\pi\, z)} = \lim_{w\to 0}\frac{w}{\sin(w)} = 1.$$

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Consequently, $\frac{\pi z}{\sin(\pi z)}$ has a holomorphic extension to $B_1(0)$ and thus a Taylor series expansion $\sum_{k=0}^{\infty} a_k z^k$ in $B_1(0)$, i.e.,

$$\frac{\pi z}{\sin(\pi z)} = \sum_{k=0}^{\infty} a_k z^k \qquad \text{for } z \in B_1(0).$$

By the above limit, the constant term is $a_0 = 1$. Using the Taylor series of sine, we have $\sin(\pi z) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\pi z)^{2\ell+1}}{(2\ell+1)!}$ and thus we obtain the coefficients a_k successively by multiplying the previous equations with (the Taylor series of) $\sin(\pi z)$:

$$\pi z = \left(\sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\pi z)^{2\ell+1}}{(2\ell+1)!} \right) \cdot \left(\sum_{k=0}^{\infty} a_k z^k \right) =$$
$$a_0 \pi z + a_1 \pi z + \pi \left(a_2 - \frac{\pi^2}{6} \right) z^3 + \pi \left(a_3 - \frac{a_1 \pi^2}{6} \right) z^4 + \pi \left(a_4 - \frac{a_2 \pi^2}{6} + \frac{a_0 \pi^4}{120} \right) z^5 + \dots$$

Comparing the left with the right side, we obtain $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{\pi^2}{6}$, $a_3 = 0$, $a_4 = \frac{7\pi^4}{360}$, ... Thus the Laurent series of $f(z)$ is

$$f(z) = \sum_{k=0}^{\infty} a_k z^{k-3} = \frac{1}{z^3} + \frac{\pi^2}{6z} + \frac{7\pi^4}{360} z + \dots,$$

and we have $\operatorname{Res}(f, 0) = \frac{\pi^2}{6}$.

As a second step, we apply the ML-inequality to the contour integral $\int_{\square_N} f \, dz$. The length of the contour is $L = 8 \cdot (N + \frac{1}{2})$. For M, we first look at[1] $g(z) = \pi / \sin(\pi z)$:

(i) $g(x + iy) = \frac{\pi}{\sin(\pi(x+iy))} = \frac{\sin(\pi x)\cosh(\pi y) - i\cos(\pi x)\sinh(\pi y)}{\sin^2(\pi x)\cosh^2(\pi y) + \cos^2(\pi x)\sinh^2(\pi y)}$

(ii) we have for the "vertical parts" of the contour

$$\left| g\left(iy \pm \left(N + \frac{1}{2} \right) \right) \right| = \left| g\left(\frac{1}{2} + iy \right) \right| = \pi \left| \frac{1}{\cosh(\pi y)} \right| = \pi \left| \frac{2}{e^{\pi y} + e^{-\pi y}} \right| \le \pi.$$

(iii) we have for the "horizontal parts" of the contour (using $|\cos x|, |\sin x| \le 1$, $\sinh y < \cosh y$ and $\sin^2(\pi x) + \cos^2(\pi x) = 1$)

$$\left| g\left(x \pm i \left(N + \frac{1}{2} \right) \right) \right| \le \pi \frac{2 \cosh\left(\pi \left(N + \frac{1}{2}\right)\right)}{\sinh^2\left(\pi\left(N + \frac{1}{2}\right)\right)} \overset{N \ge 1}{\le} 0.036\,\pi$$

Please turn over!

[1] Using $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

Therefore

$$|f(z)| \leq \frac{|g(z)|}{|z|^2} \leq \frac{\pi}{(N+\frac{1}{2})^2}$$

for $z \in \square_N = \{x+iy \mid \max\{|x|,|y|\} = N+\frac{1}{2}\}$. The ML-inequality then gives

$$\left|\int_{\square_N} f \,\mathrm{d}z\right| \leq \frac{\pi}{(N+\frac{1}{2})^2} \cdot 8 \cdot \left(N+\frac{1}{2}\right) \to 0 \qquad \text{as} \qquad N \to \infty.$$

Thus the Residue Theorem yields

$$\frac{\pi^2}{6} + 2\sum_{k=1}^{N} \frac{(-1)^k}{k^2} = \frac{1}{2\pi i}\int_{\square_N} f \,\mathrm{d}z \to 0$$

as $N \to \infty$, that is,

$$-\frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$