MA30056: Complex Analysis

EXERCISE SHEET 1: COMPLEX NUMBERS

Please hand solutions in at the lecture on Monday 2nd February.

1.) Let $z, w \in \mathbb{C}$. Verify that

(i) $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$,

(ii) $|z| = |\bar{z}|$, i.e., $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is an *isometry*,

(iii) $|z \cdot w| = |z| \cdot |w|$, i.e., $|\,.\,| : (\mathbb{C}, \cdot) \to (\mathbb{R}, \cdot)$ is a *homomorphism*,

(iv) $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.

Conclude that the *triangle inequality* $|z + w| \leq |z| + |w|$ holds.
Hint: Compute $|z + w|^2$.

Solution: Notations: we write $z = x + iy$ and $w = u + iv$ with $x, y, u, v \in \mathbb{R}$.

(i) $x^2, y^2 \leq x^2 + y^2$ for any $x, y \in \mathbb{R}$; taking square roots yields the claim.

(ii) $|\overline{z}|^2 = x^2 + (-y)^2 = x^2 + y^2 = |z|^2$ and taking sqare roots gives again the claim.
Alternatively, we have $|z|^2 = z \cdot \overline{z} = \overline{\overline{z}} \cdot \overline{z} = |\overline{z}|^2$.
Note that $\mathbb{R}^2 \ni (x, y) \mapsto (x, -y) \in \mathbb{R}^2$ is a linear map (reflection in the x-axis).

(iii) $|zw|^2 = (zw)\overline{(zw)} = z\bar{z} \cdot w\bar{w} = |z|^2|w|^2$ so that taking square roots yields the claim.

(iv) Clearly $|z| = \sqrt{x^2 + y^2} \geq 0$ by definition and $z = 0$ implies $|z|^2 = z\overline{z} = 0$.
If $z\overline{z} = |z|^2 = 0$ then $z = 0$ or $\bar{z} = 0$; but $\bar{z} = 0$ iff $z = 0$ so that $|z| = 0$ implies $z = 0$.

And now the triangle inequality:

$$\begin{aligned} |z + w|^2 &= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2 \\ &= |z|^2 + 2|z|\,|\overline{w}| + |w|^2 = |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2 \end{aligned}$$

so that taking square roots again provides the result.

2.) Show that the equation $z^n = 1$, $n \in \mathbb{N}$, has n solutions.
Determine the solutions of $z^3 = 1$.

Please turn over!

Solution: Using the polar form $z = r\,(\cos\vartheta + i\sin\vartheta)$, one has $z^n = r^n\,(\cos\vartheta + i\sin\vartheta)^n$.

- De Moivre's formula[1] states $(\cos\vartheta + i\sin\vartheta)^n = \cos(n\,\vartheta) + i\sin(n\,\vartheta)$.
- Taking the modulus in the equation $z^n = 1$ yields
$$1 = |z^n| = |z|^n = |r\,(\cos\vartheta + i\,\sin\vartheta)|^n = |r|^n\,|\cos\vartheta + i\,\sin\vartheta|^n = r^n.$$
Since we have $r \geq 0$, the only solution is $r = 1$ (independent of n).
- So, we are looking for solutions of
$$1 = (\cos\vartheta + i\sin\vartheta)^n = \cos(n\,\vartheta) + i\sin(n\,\vartheta),$$
i.e., for ϑ s.th. $\sin(n\,\vartheta) = 0$ and $\cos(n\,\vartheta) = 1$. This is the case if $n\,\vartheta = 2\pi\,k$ where $k \in \mathbb{Z}$. Therefore, solutions are given by $\vartheta = \frac{2\pi\,k}{n}$. By the 2π-periodicity of cos and sin, the n solutions of $z^n = 1$ are given by
$$\left\{\cos\frac{2\pi\,k}{n} + i\,\sin\frac{2\pi\,k}{n} \mid k \in \{0,\ldots,n-1\}\right\},$$
the *n-th roots of unity.*

For $z^3 = 1$, the solutions are 1, $-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$.

3.) Show that open/closed disks are open/closed.

Solution:

- *Open disks:* Claim: $B_r(z_0)$ is open.
Take $z \in B_r(z_0)$ and $\varrho = r - |z - z_0|$. Then $\varrho > 0$ since $|z - z_0| < r$ and, for $\zeta \in B_\varrho(z_0)$,
$$|\zeta - z_0| \leq |\zeta - z| + |z - z_0| < \varrho + |z - z_0| = r.$$
Hence $B_\varrho(z) \subset B_r(z_0)$ as required.

Please turn over!

[1] One can show $(\cos\vartheta + i\sin\vartheta)^n = \cos(n\,\vartheta) + i\sin(n\,\vartheta)$ by induction: It is clearly true for $n = 1$. For the induction step, we have
$$\begin{aligned}(\cos\vartheta + i\sin\vartheta)^{n+1} &= (\cos\vartheta + i\sin\vartheta)^n\,(\cos\vartheta + i\sin\vartheta)\\ &= (\cos(n\,\vartheta) + i\sin(n\,\vartheta))\,(\cos\vartheta + i\sin\vartheta)\\ &= \cos(n\,\vartheta)\,\cos\vartheta - \sin(n\,\vartheta)\,\sin\vartheta\\ &\qquad + i\,\sin(n\,\vartheta)\,\cos\vartheta + i\,\cos(n\,\vartheta)\,\sin\vartheta\\ &= \cos((n+1)\,\vartheta) + i\,\sin((n+1)\,\vartheta).\end{aligned}$$

- *Closed disks:* Claim: $\mathbb{C} \setminus \overline{B}_r(z_0)$ is open.
Take $z \in \mathbb{C} \setminus \overline{B}_r(z_0)$ and $\varrho = |z - z_0| - r$. Then $\varrho > 0$ since $|z - z_0| > r$ and, for $\zeta \in B_\varrho(z)$,
$$|z_0 - \zeta| \geq |z_0 - z| - |\zeta - z| > |z_0 - z| - \varrho = r.$$
Hence $B_\varrho(z) \cap B_r(z_0) = \emptyset$ as required.

4.) (*Cantor's Intersection Thm*)
Let $K_n \subset \mathbb{C}$ be compact with $K_n \supset K_{n+1}$ for all n and
$$\operatorname{diam} K_n = \sup_{z,w \in K_n} |z - w| \to 0 \qquad \text{as } n \to \infty.$$
Prove that $\exists z \in \mathbb{C} : \bigcap_{n \in \mathbb{N}} K_n = \{z\}$.

Solution: We assume that $K_n \neq \emptyset$ for all n (a usual convention says $\sup \emptyset = -\infty$ so that this would already be included in the assumption $\operatorname{diam} K_n \to 0$).

Choose $z_n \in K_n$ arbitrarily.

- $(z_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence:* Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ so that $\operatorname{diam} K_n < \varepsilon$ for any $n \geq N$.
Then, for $m, n \geq N$, $|z_m - z_n| \leq \operatorname{diam} K_N < \varepsilon$ since $K_n, K_m \subset K_N$.
- $z_n \to z$ *for some* $z \in \mathbb{C}$*:* This is because $\mathbb{C}$ is complete, i.e., because every Cauchy sequence in $\mathbb{C}$ converges.
- $\{z\} \subset \bigcap_{n \in \mathbb{N}} K_n$*:* Fix $n \in \mathbb{N}$. Then $z_m \in K_n$ for all $m \geq n$ since $K_m \subset K_n$ whenever $m \geq n$. Since K_n is closed $z = \lim_{m \to \infty} z_m \in K_n$. Therefore, $z \in K_n$ for every $n \in \mathbb{N}$, i.e., $z \in \bigcap_{n \in \mathbb{N}} K_n$.
- $\{z\} \supset \bigcap_{n \in \mathbb{N}} K_n$*:* Let $z, \tilde{z} \in \bigcap_{n \in \mathbb{N}} K_n$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $\operatorname{diam} K_n < \varepsilon$ for any $n \geq N$. Then $|z - \tilde{z}| \leq \operatorname{diam} K_N < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $|z - \tilde{z}| = 0$, i.e., $\tilde{z} = z$.

5.) Prove that $|\,.\,| : \mathbb{C} \to \mathbb{R}$ is continuous.

Solution: This is basically an application of the *inverse triangle inequality*[2]: Let $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, and choose $\delta = \varepsilon$. Then, we have for $z \in B_\varepsilon(z_0) = \{w \in \mathbb{C} \mid |w - z_0| < \varepsilon\}$ that $\varepsilon > |z - z_0| \geq ||z| - |z_0||$. Consequently, one has $|z| \in \{x \in \mathbb{R} \mid |x - |z_0|| < \varepsilon\}$ which proves the continuity at an arbitrary $z_0 \in \mathbb{C}$ and therefore on $\mathbb{C}$ (it even establishes uniform continuity on $\mathbb{C}$).

[2] The inverse triangle inequality states that for $z, w \in \mathbb{C}$ one has $||z| - |w|| \leq |z - w|$. This follows from the triangle inequality, since $|z| = |z - w + w| \leq |z - w| + |w|$ and therefore $|z| - |w| \leq |z - w|$. Interchanging the role of z and w, also yields $|w| - |z| \leq |z - w|$. This establishes $||z| - |w|| \leq |z - w|$.

Optional question:

6.) Show that $\mathbb{C}$ is *not*(!) an *ordered field*.

Note that an *ordering* of a field K is a subset $P \subset K$ having the following properties:

(O1) Given $x \in K$, we have either $x \in P$, or $x = 0$ or $-x \in P$, and these three possibilities are mutually exclusive. In other words, K is the disjoint union of P, $\{0\}$ and $-P$.

(O2) If $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$.

We shall also say that K is *ordered by* P, and we call P the set of *positive elements*.

Hint: Proof by contradiction.

Solution: Suppose, seeking a contradiction, that there exists an ordering of $\mathbb{C}$. Then we would have (as for the reals $\mathbb{R}$) $z^2 \in P$ for all $z \neq 0$: This is clear for $z \in P$, otherwise we have $-z \in P$ and applying (O2) yields $(-z)(-z) = z^2 \in P$. In particular, we would have $1^2 > 0$ and $i^2 > 0$ and thus also $0 = i^2 + 1 > 0$ which is absurd.