MA30056: Complex Analysis

Self-Assessment Sheet 10: Residues

1.) If f is holomorphic on the punctured disk $B_r^*(w)$, and $|f(z)| \geq 1$ whenenver $0 < |z-w| < \frac{1}{2}r$, then w [redacted] be an essential singularity of f.
For the black box, choose either "can" or "cannot" and justify this choice!

____

2.) Find the residues of the following functions at 0.
Click on the correspond line for the solution!

a) $z \mapsto \frac{1}{z^2}$; ____

b) $z \mapsto \frac{1}{(z-1)^2}$; ____

c) $z \mapsto \frac{\sin z}{z}$; ____

d) $z \mapsto z^2 \sin \frac{1}{z}$; ____

3.) *Fill in the blanks of the solution to the following problem!*
Problem: Evaluate the integral

$$\int_\Gamma \frac{z^2 - z + 9}{(z^2+1)(z^2+9)} \, \mathrm{d}z,$$

where Γ is the rectangle with vertices -4, 4, $4(1+i)$ and $4(-1+i)$.
Solution: The poles of the integrand are the points ____________ of which only the two points ____________ lie inside Γ. The residues at these two points are ____________ and ____________, respectively, and the value of the integral is therefore ____.

4.) *Fill in the blanks of the solution to the following problem!*
Problem: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2} \, \mathrm{d}x.$$

Solution: We consider the corresponding contour integral with integrand ________________________ with the usual closed semicircular contour Γ. The only pole inside Γ is the pole of order ____________ at the point ____________, and the residue of the function there is $\frac{1}{8i}$.
Hence, using the abbreviation $\Gamma(R) = \{R \cdot \mathrm{e}^{it} \,|t \in [0,\pi]\}$, we have $\int_{-R}^{R} \frac{x^2}{(x^2+4)^2} \, \mathrm{d}x + \int_{\Gamma(R)} \frac{z^2}{(z^2+4)^2} \, \mathrm{d}z =$ ________________________. But by the ML-inequality, there exist constants R_0 and C such that, if $R \geq R_0$, then $\int_{\Gamma(R)} \frac{z^2}{(z^2+4)^2} \, \mathrm{d}z =$ ________________________, and this ________________________ as R becomes large.
The value of the required integral is therefore ____________.