We are talking about holomorphic functions!

Theorem IV.4.1 *For a continuous function f on a domain D the following statements are equivalent:*

(i) *f is holomophic in D.*

(ii) *$f(x + iy) = u(x, y) + iv(x, y)$ with C^1-functions $u, v : D \to \mathbb{R}$ which satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$.*

(iii) *$f(z) = F(z, \overline{z})$ where F is a holomorphic function of two variables such that $F_2 = 0$ (i.e., $\frac{\partial f}{\partial \overline{z}}(z) = 0$).*

(iv) *$\int_\Gamma f \, \mathrm{d}z = 0$ for every simple closed contour Γ with $I_\Gamma \subset D$.*

(v) *f is analytic in D.*

Chapter II: Cauchy-Riemann

Theorems II.3.1 & II.3.5 (necessary & sufficient Cauchy-Riemann conditions)

$f : \mathbb{C} \to \mathbb{C}$ holomorphic $\iff$ $f : \mathbb{R}^2 \to \mathbb{R}^2$ has (continuous) partial derivatives that satisfy Cauchy-Riemann equations

Properties:

(i) conformal (Corollary II.3.2)

(ii) level curves

(iii) f constant as soon as $f' \equiv 0$ (Corollary II.3.3) or $\operatorname{Re} f$, $\operatorname{Im} f$ or $|f|$ constant

(iv) harmonic (Corollary II.3.4)

Examples:

(i) polynomials, algebra of differentiable/holomorphic functions (Theorem II.2.2)

(ii) exp, sin, cos, sinh, cosh, ... (Section II.4)

(iii) *later, after we know s.th. about anti-derivatives (& multifunctions):*
Log, arbitrary powers/roots, arccos,... (Section III.4.4)

Chapter III: Cauchy

Theorem III.2.2f. (Cauchy's Theorem(s))

f holomorphic $\Longleftrightarrow$ $\int_\Gamma f \, \mathrm{d}z = 0$, existence of anti-derivatives

<u>Path/contour integral</u> & its properties (Section III.1), then

- $\oint_{|z-z_0|=r} (z-z_0)^k = \ldots$ (Example)
- ML-inequality (Lemma III.1.1)
- FTC for path integrals (Lemma III.1.2)
- Criterion for FTC (Lemma III.2.1)

Cauchy's Theorem & Variations (Section III.2)

- simple closed contour, arbitrary domain (Theorem III.2.2)
- closed contour, simply connected domain (Corollary III.2.3)

with proofs:

- "weak versions" [f' is continuous] (Theorems III.2.5 & III.2.7)
  interchange diff. & int. or Green's Thm.
- Cauchy-Goursat [for Δ] (Theorem III.2.8)
  $\Rightarrow$ Cauchy for star-domains (Corollary III.2.10)
- Homotopy version (Theorem III.2.10)
  $\Rightarrow$ Cauchy's Formulae (Theorems III.3.1 & III.3.2)
  $\Rightarrow$ Residue Theorem (Theorem V.2.1)

Cauchy's Formulae & Applications (Section III.3 & III.4)

- derivatives of all orders (Corollary III.3.3), Morera's Theorem (Corollary III.3.4)
- using ML-inequality: Cauchy's Inequalities (Corollary III.3.5)
  $\Rightarrow$ Liouville's Theorem (Theorem III.4.1)
    $\Rightarrow$ Gauss' FTA (Theorem III.4.2)

Chapter IV: Cauchy-Taylor

Theorems IV.2.2 & IV.2.3 (Cauchy-Taylor Theorem and converse)

f holomorphic $\Longleftrightarrow$ f analytic

Convergence of power series:

- pointwise/uniform/ locally uniform/absolute convergence (Sections IV.1 & IV.2)
- Tools: Holomorphicity of locally uniform limit (Theorem IV.1.5) & Weierstrass' M-test (Theorem IV.2.1)

Uniqueness by Identity Theorem(s) (Theorems IV.3.1 & IV.3.2)
$\Rightarrow$ Maximum Modulus Theorem Theorem IV.3.3

Chapter V: Residues

What happens if f is holomorphic except for isolated singularities (poles)?

- Laurent series (Section IV.6): holomorphic in annulus (Theorem IV.6.2) & unique by Laurent's Theorem (Theorem IV.6.3)
- Laurent series yields classification of singularities
  $\Rightarrow$ Characterisation (Theorem V.1.1)
  $\Rightarrow$ Casorati-Weierstrass

- Residue Theorem (Theorem V.2.1)
  $\Rightarrow$ interesting integrals (Section V.3)
  $\Rightarrow$ interesting sums (Section V.4)