MA30056: Complex Analysis

Revision: Checklist & Previous Exam Questions II

- Define a Laurent series and define the principal part of such a series. When is a Laurent series said to converge to a function ℓ? Show that for every series

$$\sum_{k\in\mathbb{N}} a_{-k}(z-z_0)^{-k},$$

there exists $R \in [0,\infty]$ such that the series diverges for $0 < |z-z_0| < R$ and converges for $|z-z_0| > R$. Hence state the general convergence behaviour of a Laurent series.

- Given a Laurent series

$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k,$$

show that there exist $R_1, R_2 \in [0,\infty]$ such that the series converges absolutely and locally uniformly in the annulus

$$A = \{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}.$$

Hence state why functions defined by Laurent series are holomorphic in their annuli of convergence, and their derivatives are obtained by term-by-term differentiation.

- Let $f : D \to \mathbb{C}$ be holomorphic, and suppose that for some $z_0 \in \mathbb{C}$ and $R_1, R_2 \in [0,\infty]$, R_1, R_2, the annulus

$$A = \{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}$$

is such that $A \subset D$. Show that for all $z \in A$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k, \quad \text{where} \quad a_k = \frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w, \quad \forall k \in \mathbb{Z},$$

for any choice of $r \in (R_1, R_2)$.

- Define what it means for a point $z_0 \in \mathbb{C}$ to be an isolated singularity of a function $f : B \to \mathbb{C}$. Define what it means for an isolated singularity z_0 of $f : B \to \mathbb{C}$ to be (i) a removable singularity, (ii) a pole of order $n \in \mathbb{N}$, and (iii) an essential singularity.

- Suppose that $z_0 \in \mathbb{C}$ is an isolated singularity of f, so that $f : B_R^*(z_0) \to \mathbb{C}$ is holomorphic, for some $R > 0$. Show that z_0 is a removable singularity of f if and only if f is bounded in a neighbourhood of z_0, which occurs if and only if

$$\limsup_{z\to z_0} |f(z)| < +\infty;$$

z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$;

- z_0 is an essential singularity of f if and only if $\forall c \in \mathbb{C}\ \exists (z_n)_{n\in\mathbb{N}} \subset B_R^*(z_0)$ such that $z_n \to z_0$ and $f(z_n) \to c$ as $n \to \infty$.
- State the Casorati-Weierstrass theorem.
- Define what it means for a function $f : D \to \mathbb{C}$ to be meromorphic in D.
- Suppose that

  $$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

  in $B_R^*(z_0)$ for some $R > 0$. Define the residue of f at z_0. Find an alternative expression for $\mathrm{Res}(f, z_0)$ in terms of an integral, and show that if z_0 is a removable singularity of f, then $\mathrm{Res}(f, z_0) = 0$. Show that if f has a simple pole at z_0, then

  $$\mathrm{Res}(f, z_0) = \lim_{h\to z_0} (h - z_0) f(h).$$

- State the Residue Theorem. Prove the theorem by showing that the sum on the right-hand side is finite, using a contradiction argument and the compactness of $I_\Gamma \cup \Gamma$. Use the fact that the poles of f in I_Γ are isolated, and the homotopy version of Cauchy's theorem, to deduce the result.
- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define the notation

  $$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

  Find the value of

  $$\int_0^{\infty} \frac{\mathrm{d}x}{1 + x^4},$$

  by using the Residue Theorem.

8.*) `Variant of the 2005 Exam (Question 3)`

(d) Let $\Gamma := \{z = x + iy \in \mathbb{C} \mid \max\{|x|, |y|\} = 3\}$ (oriented in the anti-clockwise sense). Use the **Residue Theorem** to evaluate

$$\text{(i)} \int_\Gamma \frac{e^z\,\mathrm{d}z}{(z - (2 - i\pi/2))}, \quad \text{(ii)} \int_\Gamma \frac{e^z\,\mathrm{d}z}{2 + 4i + z}, \quad \text{(iii)} \int_\Gamma \frac{\cos(\pi z)\,\mathrm{d}z}{1 - z}.$$

11.) `2006 Exam (Question 3)`

(e) Find the Laurent series expansion of $f(z) = \frac{1}{(1+z^2)z^2}$ on the annuli

(i) $A = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ and
(ii) $B = \{z \in \mathbb{C} \mid 1 < |z|\}$.

12.) `2003 Exam (Question 4)`

(a) Determine the types of singularities at 0 possessed by the following functions f and g given by

$$\text{(i) } f(z) = z^{-1}\sin z, \quad \text{(ii) } g(z) = z\sin z^{-1}.$$

Justify your answer in each case.

13.) `2007 Exam (Question 4)`

(d) Let $n \in \mathbb{N}$ and $q = \mathrm{e}^{\frac{2\pi i}{n}}$ and define $f(z) = \sum_{k=0}^{n-1} \frac{q^k}{z-q^k}$;

(i) find the Laurent series expansion of f on a punctured disk $B^*_\varrho(q^m)$, $\varrho > 0$, for $m \in \{0, \ldots, n-1\}$ and

(ii) compute $\int_{|\zeta|=2} f \, \mathrm{d}\zeta$.

(**Hint:** $(1-q)(1+\ldots+q^{n-1}) = 1-q^n$)

14.) `2006 Exam (Question 4)`

(d) Let $\alpha \in (0, \infty) \subset \mathbb{R}$. Compute the residues of $f(z) = \frac{\mathrm{e}^{i\alpha z}}{1+z^2}$ in $\mathbb{C}$.

(e) Let $\alpha > 0$ as in (d). Show that

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2} \, \mathrm{d}x = \pi e^{-\alpha}.$$

Additional questions:

- How does $\alpha > 0$ in (e) enter?
- Why do we not compute the residues of $f(z) = \frac{\cos(\alpha z)}{1+z^2}$ in (d)?
- Can you also evaluate this integral using Cauchy's Formula (so, without explicit(!) reference to residues)?

15.) `2004 Exam (Question 4)`

(d) Use residues to evaluate $\int_0^\infty \frac{\mathrm{d}x}{1+x^6}$.

16.) `2008 Exam (Question 4)`

(d) For $n \in \mathbb{N}$, evaluate

$$\int_{|z|=1} \frac{\mathrm{e}^{(z^n)}}{z} \, \mathrm{d}z. \tag{1}$$

Hence show that

$$\int_0^{2\pi} \mathrm{e}^{\cos(n\vartheta)} \cos(\sin(n\vartheta)) \, \mathrm{d}\vartheta = 2\pi$$

and

$$\int_0^{2\pi} \mathrm{e}^{\cos(n\vartheta)} \sin(\sin(n\vartheta)) \, \mathrm{d}\vartheta = 0.$$

Additional question: Evaluate Eq. (1) using (i) residues and (ii) Cauchy's formulae.

Note: The terminology in earlier exams might be different. E.g., in the 2004 Exam, the winding number is called "index" etc.

Disclaimer: This is only a selection!! In particular, it is neither complete nor will the upcoming exam be simply a subset of these questions.
Furthermore: I will not provide answers[1] to these questions!

Good luck in the exam!

[1] Again, find them yourself! And swap yours with some other person's!