MA30056: Complex Analysis

Revision: Checklist & Previous Exam Questions I

- Given $z \in \mathbb{C}$ and $r > 0$, define $B_r(z)$ and $\overline{B}_r(z)$.
- Define what it means for a subset $A \subset \mathbb{C}$ to be open/closed.
  If $M \subset A \subset \mathbb{C}$, when is M said to be open/closed in A?
- Define what it means for a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ to be convergent.
  State the Bolzano-Weierstrass and Heine-Borel theorems, and state a result concerning the completeness of $(\mathbb{C}, |\cdot|)$.
- Define a path, and define what it means for a path $\gamma : [a, b] \to \mathbb{C}$ to be simple, closed and simple-closed, respectively.
  When is $\Gamma \subset \mathbb{C}$ said to be a (simple/closed/simple-closed) Jordan curve?
- When is $A \subset \mathbb{C}$ said to be path-connected?
  Define a domain $D \subset \mathbb{C}$.
  State the Jordan curve theorem.
  When is a domain $D \subset \mathbb{C}$ said to be simply connected?
- Show that if $\gamma : [a, b] \to \mathbb{C}$ is a path and $[\alpha, \beta] \subset [a, b]$, then $\gamma|_{[\alpha,\beta]} : [\alpha, \beta] \to \mathbb{C}$ is also a path.
  Define the composition of two paths and show that it is a path.
  Define the inverse of a path.
  When is a path $\gamma : [a, b] \to \mathbb{C}$ said to be smooth?
  When is a path $\gamma : [a, b] \to \mathbb{C}$ said to be regular/piecewise smooth/piecewise regular?
  When is $\Gamma \subset \mathbb{C}$ said to be a (simple/closed/simple-closed) contour?
  Let $\gamma : [a, b] \to \mathbb{C}$ be a regular path and $\Gamma = \gamma([a, b])$. If $h : [\tilde{a}, \tilde{b}] \to [a, b]$ is surjective and regular, show that $\tilde{\gamma} := \gamma \circ h : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is a new regular parametrisation of Γ.
  If $\gamma : [a, b] \to \mathbb{C}$ is a regular path, define the length of $\Gamma = \gamma([a, b])$. Hence define the length of a contour.
- Define what it means for a function $f : D \to \mathbb{C}$ to be continuous on $D \subset \mathbb{C}$. Define what it means for $f : \mathbb{C} \supset D \to \mathbb{C}$ to be uniformly continuous on D.
- Let $f : D \to \mathbb{C}$ be defined on a domain $D \subset \mathbb{C}$. Define what it means for f to be complex differentiable at $z \in D$. Define what it means for h to complex differentiable/holomorphic on D.
- Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic and write $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Show that the first-order partial derivatives u_x, u_y, v_x and v_y of u and v exist on D and satisfy $u_x = v_y$, $u_y = -v_x$ in D, and that

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z), \quad \forall z \in D.$$

- Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic in D with $f' \equiv 0$ in D. Show that f is constant on D.

- Let $f = u + iv : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic. Show that $u, v : D \to \mathbb{R}$ are harmonic functions and that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla u = \nabla v \quad \text{in } D.$$

- Let $u, v : \mathbb{R}^2 \supset D \to \mathbb{R}$ be continuously differentiable on D and satisfy $u_x = v_y$, $u_y = -v_x$ on D. Show that $f = u + iv : D \to \mathbb{C}$ is holomorphic, using the fact that f must be real differentiable in D, viewed as a subset of $\mathbb{R}^2$ or of $\mathbb{C}$ as convenience dictates.

1.) `2007 Exam (Question 1)`

(c) Let $\varphi : D \to \mathbb{R}$ be twice continuously differentiable and harmonic. Show that $f = \varphi_x - i\,\varphi_y$ is holomorphic.

(d) Show that every harmonic function $\varphi : D \to \mathbb{R}$, where D is simply connected, is the real part of a holomorphic function.
For your proof **you may assume** that:

(i) every holomorphic function $f : D \to \mathbb{C}$ has an anti-derivative $F : D \to \mathbb{C}$;
(ii) if $u_x = \varphi_x$ and $u_y = \varphi_y$ for $u, \varphi : D \to \mathbb{R}$, then $\varphi = u + c$ for some $c \in \mathbb{R}$.

(e) Let $\varphi(x, y) = x^2 - y^2$; verify that φ is harmonic and find a holomorphic function F with $\varphi = \operatorname{Re} F$.

2.) `2005 Exam (Question 1)`

(d) Let $u(x, y) = x\,y$. Find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ with $u = \operatorname{Re} f$.

3.) `2008 Exam (Question 1)`

(f) Let $u(x, y) = \sin(x)\cos(y)$. Is there a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ with $u = \operatorname{Re} f$? If so, say which function f; if not, give a reason.

- Let $f : D \to \mathbb{C}$ be continuous and let $\gamma : [a, b] \to D$ be a piecewise regular path. Define the path integral

$$\int_\gamma f(z)\,\mathrm{d}z,$$

and state why this is well-defined.

- Show that the path integral has the property of parameter invariance: that is, show that if $\gamma : [a, b] \to D$ and $\tilde{\gamma} = \gamma \circ h : [\tilde{a}, \tilde{b}] \to D$ are two parametrisations of a contour $\Gamma = \gamma([a, b])$, then

$$\int_{\tilde{\gamma}} f(z)\,\mathrm{d}z = \pm \int_\gamma f(z)\,\mathrm{d}z.$$

What is the relationship between

$$\int_{-\gamma} f(z)\,\mathrm{d}z \quad \text{and} \quad \int_{\gamma} f(z)\,\mathrm{d}z?$$

- Read the subsection on the orientation of a contour.
- Let $f : \gamma([a,b]) \to \mathbb{C}$ be continuous, and denote

$$M = \sup_{z\in\gamma([a,b])} |\, f(z)\,|, \quad L = \int_a^b |\,\gamma'(t)\,|\,\mathrm{d}t.$$

  Show that $M < \infty$ and

$$\left|\int_{\Gamma} f(z)\,\mathrm{d}z\right| \leq ML.$$

- Let $f : D \to \mathbb{C}$ be continuous on D and suppose that f has an anti-derivative $F : D \to \mathbb{C}$ (so that F is holomorphic in D with $F' = f$). Show that
  - for any piecewise regular path $\gamma : [a,b] \to D$, it follows that

$$\int_{\gamma} f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)),$$

    and
  - for every closed contour $\Gamma \subset D$, it follows that

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0.$$

- Let $f : D \to \mathbb{C}$ be holomorphic in a domain $D \subset \mathbb{C}$ and let $\Gamma \subset D$ be a simple closed contour such that $I_\Gamma \subset D$. State Cauchy's theorem for f and Γ.
- State Cauchy's theorem for simply connected domains.
- Let $f : D \to \mathbb{C}$ be continuous on a domain D and assume that

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0,$$

  for every closed contour $\Gamma \subset D$. Show that f has an anti-derivative $F : D \to \mathbb{C}$. Hence, show that if f is holomorphic on a simply connected domain D, then f has an anti-derivative $F : D \to \mathbb{C}$.
- State a weak version of Cauchy's theorem. Prove this version of Cauchy's theorem by using Green's Theorem.
- Define a triangle and state the Cauchy-Goursat theorem.
- Define a star-domain. Prove if $f : D \to \mathbb{C}$ is holomorphic on a star-domain $D \subset \mathbb{C}$, then f has an anti-derivative $F : D \to \mathbb{C}$.
- State and prove Cauchy's theorem for star-domains.

- State the Homotopy Version of Cauchy's theorem, and give a generalisation of this theorem to multiple simple closed contours.
- Evaluate
$$\int_{|z|=2} \frac{\mathrm{d}z}{z^2-1},$$
by using the generalisation of the Homotopy Version of Cauchy's theorem.
- Let $f : D \to \mathbb{C}$ be holomorphic on a domain D and let $\Gamma \subset D$ be a simple closed contour such that $I_\Gamma \subset D$. Show that, for all $z \in I_\Gamma$,
$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw.$$
- Let $f : D \to \mathbb{C}$ be holomorphic and let $\Gamma \subset D$ be a simple closed contour such that $I_\Gamma \subset D$. Show that
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw, \quad \forall z \in I_\Gamma, \quad \forall n \in \mathbb{N}_0.$$
(State any result you are using.)
- State Morera's theorem.
- Let $f : D \to \mathbb{C}$ be holomorphic and suppose that, for $z_0 \in \mathbb{C}$, the circle $\partial B_R(z_0)$ is such that $B_R(z_0) \subset D$. Show that for any $n \in \mathbb{N}$,
$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n}\max_{z\in\Gamma}|f(z)|.$$
- Define an entire function, and show that any bounded entire function is constant (Liouville's Theorem). State Gauss' Fundamental Theorem of Algebra.
- Let $f : B_R(z_0) \to \mathbb{C}$ be holomorphic, and assume that
$$|f(z)| \leq |f(z_0)|, \quad \forall z \in B_R(z_0).$$
Show that f is constant in the disk $B_R(z_0)$.
- What is a logarithm on a simply connected domain D?
What is the principal value (of the logarithm) $\mathrm{Log}(z)$ of the number z?
Define the power a^b for complex numbers a, b.

4.) `2007 Exam (Question 2)`

(d) Let $f : D \to \mathbb{C}$ be holomorphic and $r > 0$ so that the closed disk $\overline{B}_r(z_0) \subset D$; assume that $\min_{|z-z_0|=r}|f(z)| > |f(z_0)| > 0$. Show that f has a zero in $B_r(z_0)$.
(**Hint:** suppose not and consider $g(z) = \frac{1}{f(z)}$).

5.) `2005 Exam (Question 4)`

(e) Let f be an entire function and suppose that $|f(z)| \le e^{\operatorname{Re} z}$ for all $z \in \mathbb{C}$. Prove that there exists a $c \in \mathbb{C}$ with $|c| \le 1$, such that $f(z) = c\, e^z$ for all $z \in \mathbb{C}$.

6.) `2008 Exam (Question 3)`

(e) Show that $f(z) = e^{i|z|}$ is not an entire function. Justify all your claims.

(f) Suppose that f is entire and that $|f(z)| \le |z|^2$ for all sufficiently large values of $|z|$, say, for $|z| \ge r$. Prove that f must be a polynomial of degree at most 2.
(**Hint:** Use *Cauchy's Inequalities for the Derivatives* to conclude that $f^{(3)}$ vanishes.)

7.) `2005 Exam (Question 2)`

(c) Let $f(z) = \frac{1}{1+z^2}$.
 (i) Show that there is a closed regular path $\gamma : [a,b] \to \mathbb{C} \setminus \{\pm i\}$ with $\int_\gamma f(z)\,dz \ne 0$.
 (ii) Deduce that f cannot have an anti-derivative in $\mathbb{C} \setminus \{\pm i\}$.

(d) Let $f(z) = 4z^4$ and $a > b > 0$. Compute the path integral $\int_\gamma f(z)\,dz$, where

$$\gamma : [0,1] \to \mathbb{C}, \quad t \mapsto \gamma(t) = a \sin\frac{\pi t}{2} + ib\left(1 - \cos\frac{\pi t}{2}\right).$$

Additional question: What is the catch in (d)?

8.) `2005 Exam (Question 3)`

(d) Let $\Gamma := \{z = x + iy \in \mathbb{C} \mid \max\{|x|, |y|\} = 3\}$ (oriented in the anti-clockwise sense). Evaluate

$$\text{(i)} \int_\Gamma \frac{e^z\,dz}{(z - (2 - i\pi/2))}, \quad \text{(ii)} \int_\Gamma \frac{e^z\,dz}{2 + 4i + z}, \quad \text{(iii)} \int_\Gamma \frac{\cos(\pi z)\,dz}{1 - z}.$$

9.) `2008 Exam (Question 2)`

(c) Show:
 (i) $f(z) = \frac{1}{z}$ has an anti-derivative in the cut plane $\mathbb{C} \setminus \mathbb{R}_{\le 0}$, but
 (ii) $f(z) = \frac{1}{z}$ has no anti-derivative in the punctured plane $\mathbb{C} \setminus \{0\}$.
 State any result you are using to establish these claims.

(e) Is there a logarithm of 0? Justify your answer.

(f) Calculate all possible values of i^i. What is the principal value of i^i?

- Define what it means for the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ (where $f_n : D \to \mathbb{C}$) to converge uniformly to the function $f : D \to \mathbb{C}$ on D.
  State the Cauchy criterion for uniform convergence, in this context.
  Define what it means for a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ to converge locally uniformly to a function f on D.
  Show that the locally uniform limit of continuous functions is continuous.
- Show that if $\gamma : [a, b] \to \mathbb{C}$ is piecewise regular, $\varGamma = \gamma([a, b])$, and $f : \varGamma \to \mathbb{C}$ is the uniform limit of a sequence of continuous functions $f_n : \varGamma \to \mathbb{C}$, then

$$\int_\varGamma f(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_\varGamma f_n(z)\,\mathrm{d}z.$$

- Let $\{f_n\}_{n\in\mathbb{N}}$ (where $f_n : D \to \mathbb{C}$ for all n) be a sequence of holomorphic functions converging locally uniformly to a function $f : D \to \mathbb{C}$ on D. Show that f is holomorphic, and that $f_n' \to f'$ locally uniformly on D, as $n \to \infty$.
- State the Weierstrass M-test for a sequence of functions $\{f_k\}_{k\in\mathbb{N}_0}$.
- Show that functions defined by power series are holomorphic within the disk of convergence, and that the derivative is obtained by term-by-term differentiation.
- Let $f : D \to \mathbb{C}$ be holomorphic and suppose that $B_R(z_0) \subset D$ for some $z_0 \in D$ and $R > 0$. Show that for any $z \in B_R(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k, \quad \text{where } a_k = \frac{f^{(k)}(z_0)}{k!}, \quad \forall k \in \mathbb{N}_0.$$

- Define what it means for a function $f : D \to \mathbb{C}$ to be analytic in D.
- Let $f(z) = \sum\limits_{k=0}^{\infty} a_k(z - z_0)^k$ be a power series with radius of convergence $R > 0$ (or $R = \infty$). Suppose that $f(z_n) = 0$, $\forall n \in \mathbb{N}$, for some sequence $(z_n)_{n\in\mathbb{N}} \subset B_R(z_0) \setminus \{z_0\}$ such that $z_n \to z_0$ as $n \to \infty$. Show, by induction, that $a_k = 0$, $\forall k \in \mathbb{N}_0$ (that is, $f \equiv 0$).
  Hence show that the power series expansion of a function is unique.
- Suppose that $f : D \to \mathbb{C}$ is holomorphic and satisfies $f(z_n) = 0$, $\forall n \in \mathbb{N}$, for some sequence $(z_n)_{n\in\mathbb{N}} \subset D \setminus \{z_0\}$ such that $z_n \to z_0 \in D$ as $n \to \infty$. Show that $f \equiv 0$.
  Hint: Start by defining N and N'. Next, show that $\varnothing \neq N' \subset N$; that N' is open and closed in D and deduce that $N' = D$.
- Let $f : D \to \mathbb{C}$ be holomorphic and suppose that $|f| : D \to \mathbb{R}$ has a maximum $z_0 \in D$. Show that f is constant.

10.) One[1] of your suggestions, see `Question 6 on Exercise sheet 6`.

(a) Define $\int_\gamma f(z)\,\mathrm{d}z$ for f continuous and $\gamma : [a, b] \to D$ a piecewise regular path.

[1]Well, the only one!

(b) State the three properties of a path integral (no proof required).

(c) State and prove the ML-inequality.

(d) Let $\Gamma \subset \mathbb{C}$ be a simple closed contour and $z_0 \in I_\Gamma$ a point in its interior. Prove that there is a $\varrho > 0$ s.t.

$$\left| \int_\Gamma \frac{\mathrm{d}z}{(z - z_0)^k} \right| < \frac{1}{\varrho^k} L.$$

Disclaimer: This is only selection!! In particular, it is neither complete nor will the upcoming exam be simply a subset of these questions.
Furthermore: I will not provide answers[2] to these questions!

[2]Find them yourself! And swap yours with some other person's!