University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30056: Complex Analysis

Thursday 21st May 2009, 16.30–18.30

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. Let $D \subset \mathbb{C}$ be a domain.

(a) What does it mean to say that a function $f : D \to \mathbb{C}$

(i) is *complex differentiable* at $z \in D$?

(ii) is *holomorphic*? [4]

(b) For a function $f : D \to \mathbb{C}$, we write $f(x + iy) = u(x, y) + iv(x, y)$ with $x, y \in \mathbb{R}$ and real-valued functions u and v. State:

(i) the *necessary Cauchy-Riemann conditions* (for f to be holomorphic).

(ii) the *sufficient Cauchy-Riemann conditions* (for f to be holomorphic). [4]

(c) Give an example of a continuous complex function $f : \mathbb{C} \to \mathbb{C}$

(i) that is nowhere complex differentiable.

(ii) that is complex differentiable at only one point.

In each case, prove your claim! [4]

(d) Let $u(x, y) = x^2 + 2a\,x\,y + b\,y^2$ where $a, b \in \mathbb{R}$.
Determine all values (a, b) for which u is the real part of a holomorphic function. Hence express these holomorphic functions as polynomials in z. [3]

(e) State and prove the *ML-inequality*. [5]

2. (a) (i) What is a *simple closed contour* $\mathit{\Gamma} \subset \mathbb{C}$?

(ii) Define the *path integral* $\int_\gamma f(z)\,\mathrm{d}z$ for a continuous function $f : D \to \mathbb{C}$ and a piecewise regular path $\gamma : [a, b] \to D$. [4]

(b) State and prove a criterion that justifies the use of anti-derivatives in evaluating path integrals. [7]

(c) State *Cauchy's Theorem.* [2]

(d) Evaluate the following integral. Give your reasoning.

$$\int_{\mathit{\Gamma}} (2z^2 \cos(z) \sinh(z) - 3\,\overline{z})\,\mathrm{d}z,$$

where $\mathit{\Gamma} = \partial B_2(5)$ is the circle of radius 2 around 5. [3]

(e) (i) Use (b) to write down the value of

$$\int_0^{a+ib} \mathrm{e}^z\,\mathrm{d}z.$$

(ii) Equate the previous answer with the one obtained by parametric evaluation along the straight contour from 0 to $(a + ib)$, and deduce

$$\int_0^1 \mathrm{e}^{ax} \cos(bx)\,\mathrm{d}x = \frac{a\,(\mathrm{e}^a \cos b - 1) + b\ \mathrm{e}^a\ \sin b}{a^2 + b^2}, \quad \text{and}$$
$$\int_0^1 \mathrm{e}^{ax} \sin(bx)\,\mathrm{d}x = \frac{b\,(1 - \mathrm{e}^a \cos b) + a\ \mathrm{e}^a\ \sin b}{a^2 + b^2}.$$

[4]

3. (a) State *Gauss' Fundamental Theorem of Algebra.* [2]

(b) State the *Maximum Modulus Theorem.* [2]

(c) Prove the following corollary:
If $f(z)$ is holomorphic in $B_R(0)$ for some $R > 0$ and continuous in $\overline{B}_R(0) = \{z \in \mathbb{C} \mid |z| \leq R\}$, then $|f(z)|$ takes on its maximum on the boundary $\partial B_R(0)$ (the circle of radius R). [3]

(d) (i) Find a function f that is holomorphic on the unit disk $B_1(0)$ and continuous on $\overline{B}_1(0)$ that realises the minimum of $|f(z)|$ in the interior $B_1(0)$ but not on the boundary $\partial B_1(0)$.
(In other words: show – using a counterexample – that the statement in (c) does not hold if you replace "maximum" by "minimum".)

(ii) Let $f(z) = \mathrm{e}^z$. Calculate the maximum and minimum values of $|f(z)|$ on $\overline{B}_1(0)$ and show that they occur on the boundary of $\partial B_1(0)$.

(iii) Prove the following *Minimum Modulus Principle*:
Let f satisfy the assumptions of the corollary in (c) and assume that $f(z) \neq 0$ for all $z \in \overline{B}_R(0)$. Then $|f(z)|$ takes on its minimum on the boundary $\partial B_R(0)$. [7]

(e) Let $p(z) = \sum_{k=0}^{n} a_k \cdot z^k$ (where $n \geq 1$ and $a_n \neq 0$) be a polynomial.
Show: $|p(z)| \to \infty$ as $|z| \to \infty$.
(**Hint:** (inverse) triangle inequality.) [3]

(f) Prove *Gauss' Fundamental Theorem of Algebra* **using** the *Minimum Modulus Principle.* [3]

4. (a) (i) Give the definition of a *Laurent series*.

(ii) Give the definition of the *residue* $\operatorname{Res}(f, z_0)$ of a function f at $z_0 \in \mathbb{C}$.

(iii) Show: If f has a simple pole at z_0, then $\operatorname{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0)f(z)$. [6]

(b) State the *Residue Theorem*. [2]

The *n-th Fibonacci number* f_n, where $n \geq 0$, is defined by the following recurrence relation:

$$f_0 = 1, \quad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

Let $F(z) = \sum_{n=0}^{\infty} f_n \cdot z^n$.

(c) Prove (by induction): $f_n \leq 2^n$ for all $n \in \mathbb{N}$.
Hence deduce that the radius of convergence of F is greater than or equal to $\frac{1}{2}$. [3]

(d) Show that the recurrence relation among the f_n implies that $F(z) = \frac{1}{1-z-z^2}$.
(**Hint:** Write down the power series of $z \cdot F(z)$ and $z^2 \cdot F(z)$ and rearrange both so that you can easily add.) [3]

(e) Verify that $\operatorname{Res}\left(\frac{1}{z^{n+1}\cdot(1-z-z^2)}, 0\right) = f_n$. [2]

(f) Use the Residue Theorem to derive the explicit formula

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

(**Hint:** Integrate $\frac{1}{z^{n+1}\cdot(1-z-z^2)}$ around a circle with centre 0 and radius R and show that this integral vanishes as $R \to \infty$.) [4]