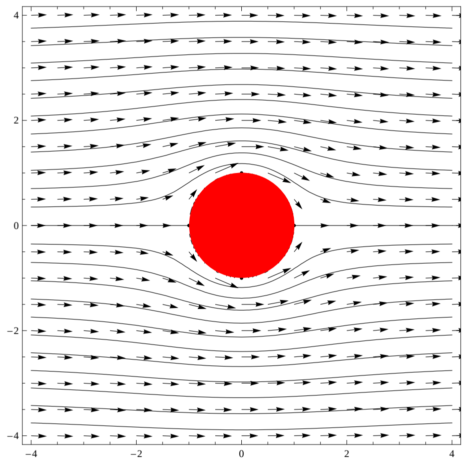
MA30056: Complex Analysis

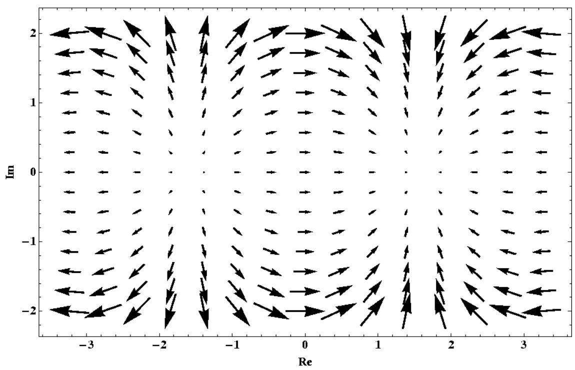
III.4.3. Physical Applications (Fluid dynamics) (Not examinable!)

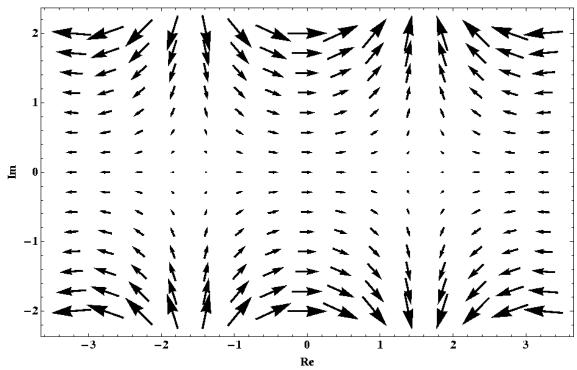
We look at harmonic functions again (compare Corollary II.3.4), and first show why they are studied in physics (fluid dynamics/electrostatics):

A complex function $\tilde{f}(z)$ defines a two-dimensional vector field $(u, v, 0)$ which represents a steady-state (there is no time-dependence), laminar (stratified into layers[1]), incompressible (in mathematical terms: $\operatorname{div} \tilde{f} = \nabla \cdot \tilde{f} = \tilde{u}_x + \tilde{v}_y = 0$) and irrotational (in mathematical terms: $\operatorname{curl} \tilde{f} = \nabla \times \tilde{f} = (\tilde{v}_x - \tilde{u}_y) e_z = 0$; there are no vortices ("whirlpools")) fluid flow over a domain D precisely when the conjugate of $\tilde{f}(z)$ is holomorphic. In electrostatics, $\tilde{f}(z)$ describes the electrical vector field in a charge-free domain D (or the charge-free part of the domain D).



In Section II.4, we have already compared the vector fields for for $f(z) = \cos(z)$ (see below on the left) and $f(z) = \overline{\cos(z)}$ (see below on the right). Here, we note that the vector field on the right satisfies the conditions just stated (e.g., it has no "sinks" and "springs" where the incompressibility assumptions would certainly not hold).





Remark. For a holomorphic function f, if we would assume continuity of f', then Poincaré's Lemma would yield an alternative proof for the existence of an anti-derivative of f.

Poincaré's Lemma. Let $\vec{v} = (\alpha, \beta) : D \to \mathbb{R}^2$ be a continuously differentiable vector field on a star-domain D such that $\alpha_y = \beta_x$ (i.e., $\operatorname{curl} \vec{v} = 0$). Then $\vec{v}$ has a *potential* $\varphi : D \to \mathbb{R}$, i.e., φ is (real) differentiable with $\nabla \varphi = \vec{v}$.

Lemma. If $f : D \to \mathbb{C}$ is holomorphic with continuous f' in a star-domain D, then it

[1] If you want to see what the effect of "laminar" is, look at the following physics video:

http://www.youtube.com/watch?v=p08_KlTKP50 has an anti-derivative $F : D \to \mathbb{C}$.

Proof. Write $f = u + iv$; we are seeking $F = U + iV$ so that

$$F' = U_x + iV_x = V_y - iU_y = u + iv \qquad \Leftrightarrow \qquad \begin{cases} \nabla U = (u, -v) & \text{and} \\ \nabla V = (v, u) \end{cases}$$

Now, since f is holomorphic,

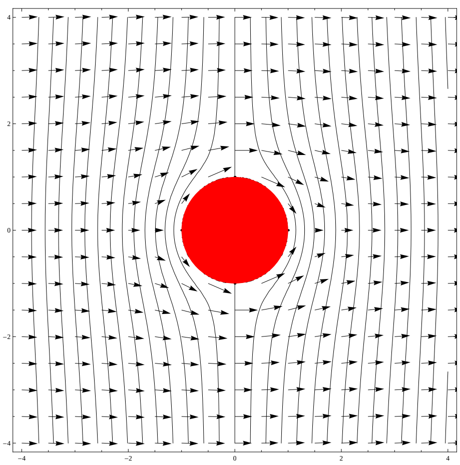
$$\begin{aligned} u_y = -v_x \qquad &\Rightarrow \qquad \exists U : D \to \mathbb{R} : \nabla U = (u, -v) \\ v_y = u_x \qquad &\Rightarrow \qquad \exists V : D \to \mathbb{R} : \nabla V = (v, u) \end{aligned}$$

by Poincaré's Lemma since the partial derivatives of u and v are continuous and D is a star-domain.

Finally, U and V are by their gradients determined up to a real constant each. These constants combine to determine F up to a complex constant. □

In physics, a vector field that possesses a potential is called *conservative* or *exact*. In that case, the work exerted to move a particle along a path, only depends on the endpoints of the path (the difference of the potential at the endpoints).

Also note that the functions U and V in the previous proof are harmonic conjugate to each other. In the picture at the beginning of this section, we showed a vector field of a fluid flowing around a unit disk. In that picture, the *stream-lines* are shown (which are the level curves $V(x, y) = \text{const.}$), i.e., the paths along which the fluid (or "fluid particle") flows. On the figure on the right, we depicted the *equipotential lines* (the level curves $U(x, y) = \text{const.}$) which shows us how the fluid is advancing in each unit of time.



We can also use Poincaré's Lemma to prove a converse of Corollary II.3.4.

Lemma. Let $u : D \to \mathbb{R}$ be a twice continuously (partial) differentiable harmonic function on a star-domain D. Then u is the real part of a holomorphic function.

Proof. We let $\alpha = -u_y$ and $\beta = u_x$. Then α and β are continuously differentiable with $\beta_y - \alpha_x = \Delta u = 0$ so that, by Poincaré's Lemma, there is $v : D \to \mathbb{R}$ with $v_x = \alpha = -u_y$ and $v_y = \beta = u_x$.

Now, as u and v are both continuously differentiable and satisfy the Cauchy-Riemann equations $f = u + i\,v$ is holomorphic by the sufficient Cauchy-Riemann conditions. □

ex Let $u(x, y) = xy$ on $D = \mathbb{C}$. Show that u is harmonic and find a holomorphic function f with $\operatorname{Re} f = u$.

Solution: Clearly $\Delta u = u_{xx} + u_{yy} = 0 + 0 = 0$ and observe that $u(x, y) = -\operatorname{Re}(\frac{iz^2}{2} - 42i)$.