MA30056: Complex Analysis

FEEDBACK: EVALUATION & FINAL EXAM

Remarks on the Course Evaluation

General ("the bare numbers") Responses: 36 (out of 87 $\cong$ 41%).

Question	mean[1]	median	low. &	upp. quart.
My understanding of the subject is increasing.	4.00	4	4	5
I find the teaching methods used are effective.	3.83	4	3	4
Sufficient advice and support is available to me.	4.19	4	4	5
Overall, I am satisfied with the quality of this unit.	3.86	4	3	4
The main ideas were communicated clearly.	3.86	4	3	4
It was easy to make good notes.	4.25	4	4	5
The lectures were well presented.	3.89	4	4	4.5
The subject matter was interesting.	3.61	4	3	4
Was the pace of the lecture right for you.	3.39	3	3	4
Was the level of material right for you?	3.44	3	3	4

I am really very delighted with these results, compared to last years' evaluation in 'Complex Analysis' (when I taught it for the first time) I improved almost in all points (in particular, this year I managed to do justice to the 'interesting' material, previously the response to the eight question was 'neutral') with highlights in the 'good notes', 'advice & support' and 'understanding'-question. Thank you!

I only seem to have been faster (a bit too fast at times(?)) this year (might be true, actually, hmm). Other than that, the many 'agree' and 'strongly agree' are great[2]!

[1] I hope that you all learnt in statistics that it is dubious to calculate the mean of ordinal data, although it is done in these unit evaluations – so, I included lower and upper quartiles in addition to the median to get a more scientifically justified representation of your responses to this five-point likert scale questions.

[2] O.K., there are always some strange lunatic responses: three people claim that their understanding of the subject has decreased (they 'disagree that their understanding has increased') and one person finds the teaching methods totally ineffective ('strongly disagree that they are effective') and claims that the presentation of the lectures was a total mess ('strongly disagree that lectures are well presented'). If you are that negative at least bother to add some explaining comments preferably already during the semester, otherwise one cannot take you seriously (I don't, but wish you good luck with that scholarship to Sandhurst[3]).

[3] 'You're fired!'

What do I, the lecturer, do well (in your opinion): Thanks for the nice and encouraging comments which include:

- *"Clear notes and well presented and also lots of resources online"*
- *"Is really enthusiastic about the course/lecturing. Gives extra examples/pictures."*
- *"Tries to engage with the audience and get feedback throughout the term. Always cheerful. Self assessment sheets useful for revision and I'm hoping the revision handouts will be too. Interactive problems class good although not much time to make notes."*
- *"Has a drop in so he can answer questions about the week's lecture more clearly than in a general lecture. He sees a picture of puzzlement on someone's face and immediately explains why something happens if he understands the bit you don't get."*
- *"– He offers help all the time. – He is VERY (!) dedicated. – He uses animations, pictures and videos to visualise main concepts. – His lecture notes are of great help. (There is always a reference to text books at the beginning of section – very good) – He offers a drop in session. – He asks all the time for feedback and what he can improve. – He wants his students to perform well. – There was even a social event (watching a movie) for those who were interested! – He answers every question."*

  Okay, so (again, as after M41!) nobody mentioned the chocolates. . . ;-)[4]

What do I not do so well (in your opinion):

- *"There was far too much material in the course to cover and we often over-ran in the lectures. Also it was hard trying to revise when we still had lots of teaching material to cover in the last teaching week."*
- *"Too many problem sheets! Runs into revision week – has self assessment sheets too which I find with all the material is hard to keep on top of, including having 2 revision sheets (helpful but no answers)."*

  I totally agree that my over-running habit this semester was awful (sorry). I don't agree with the criticisms about revision – the first revision sheet was handed-out at 16 March and covered everything up to the Easter break (so, the bulk of this unit except Laurent series and residues). Thus, there was more than one month (lectures restarted at 20 April after the Easter break) to revise and try to understand everything up to the material of the last two weeks – Question 6 on Exercise sheet 8 (due on 20 April!) even explicitly asked if something is still unclear in the material covered so far! And that there are no answers on the revision sheets – see my

[4] Raised in Upper Swabia, did my PhD in East-Westphalia – in these regions in Germany, even if your local football team is up 5–1 in the 85th minute once in decade, you say: well, okay, but even now they don't play as exceptional as Barça!

comment on page 7 in the FAQs on my intention that the revision sheets should help you to condense the material in this unit. That comment shows some "consumer mentality" that also becomes apparent in the following one.

- *"There is too much material in the course for the number of lectures which hampers understanding and meant the lecturer often had to rush sections and give handouts and then you don't understand thoses sections as well.* [...] *Also in problem classes I would have preferred him just to go through all or as many of the questions on the problem sheets as possible rather than asking the audience which questions they want him to go through."*

  Usually, handouts contained additional non-examinable material for the interested students (the psychologically unfortunate exceptions occured in the last lecture: the applications of the residue theorem for real integrals and $\zeta(2)$ where I used handouts to distract you from some details of the calculations involved but rather wanted to concentrate on the overall method; of course, there is always this time-limit at 16:07 plus-minus some minutes...). The last part of this comment is only allowed if you belong to the around 8 students that handed-in solutions every week.

- *"Sometimes proofs were rushed or unclear."*
- *"Perhaps to go through the proofs in a bit more details and to explain why things are being done in the proofs instead of leaving so many things as "exercises".*
- *"explain more in between the theorems"*

  I admit that I do some of the typical mistakes one does at the beginning of one's lecturer career: I sometimes want to cover and mention too much and then don't spend enough time to explain the basics (the 'that's clear/trivial, isn't'-effect). Here, I am grateful to the people that showed up in the drop-in sessions and asked questions and to those whose questions you can read in the FAQs – such questions are a good indicator where I spent too little time in the lecture.

- *"Give more examples in the lecture notes so that it's possible to do the problem sheets. A proper explanation of winding numbers."*

  More examples in the lecture NOTES! Seriously? I am confused that seems to contradict some of the other comments.

- *"Perhaps split lecture into 2 1 hour lectures."*

  I will think about that (although it is too late for this year now).

- *"no not really, make the course easier? The only thing the lecturer should be careful of is using sarcasm in his marking – it can be quite disheartening to read when you're struggling with the module."*

  I really apologize if I was sarcastic in any way. That was not my intention, in particular not towards people who do the exercise sheets!

Other comments:

- *"There appears to be a lot more material in this course than any other course, coupled with difficult topics makes this a very demanding module. I would also note that in the revision lecture many students were surprised at the size of exam questions – in comparison to their other units – and also "harsh" mark scheme."*

  On the first part, compare my remark above. I certainly still have to learn to lecture with more superior ease, omit some of the distracting details (which make some things looking harder than they actually hard) but explain more the key points – I hope this comes with experience!

  On the revision lecture: I rather pretend to have a harsh marking scheme and in reality be a soft marker (you would be surprised, honestly!).

- *"The keypad system (I'm afraid I can't remember the product's name) is a fun and enjoyable test knowledge, in an anonymous manner. The department should use them in a wider range of units."*

  Here in Bath it is called ARS (Audience Response System), at other universities they rather call it EVS (Electronic Voting System); no points here if you figure out why!

- *"Great unit. Love the material. Maybe the Residue Calculus stuff could be left as unexaminable as this is a hard chapter to grasp ever so close to the exams."*

  The calender certainly played an unfortunate role with this two weeks of teaching after the Easter break and just before the exam (too long to do only non-examinable things, too short to let examinable things sink in). And would you have bothered about non-examinable material?

- *"The best lecturer I had this year! He really prepares his classes and is concerned with the students understanding the material. He is also very enthusiastic and tries to transmit this to the class. Keep it up!*

  (lecturer simply speechless)

Now, they say the proof of the pudding is in the eating. So, let us now look at the final exam and who well I taught understanding of the material!

Remarks on the Final Exam I first look at all the questions individually before giving some general remarks. I always start with some statistics on each question (mean, median etc. are always calculated w.r.t. actual attempts of this part of the question). Unfortunately, I am not allowed to provide the model solutions as pdf-document at this stage, so either you have to wait until they are available at the library or you can always ask me directly. Also shown are some actual examples from your exams (• "...").

Question 1

Question	maximal marks	mean	standard dev.	median	no. of attempts
(a)(i)	2	1.92	0.38	2	87
(a)(ii)	2	1.99	0.11	2	87
(b)(i)	2	1.93	0.25	2	87
(b)(ii)	2	1.79	0.43	2	87
(c)(i)	2	1.06	0.91	1	67
(c)(ii)	2	0.69	0.87	0	52
(d)	3	1.23	0.93	1	78
(e)(i)	2	1.63	0.59	2	87
(e)(ii)	3	2.38	0.89	3	85
overall	20	13.92	2.78	14	87

(a) Most people got these fundamental definitions of this unit right (they can be found in Section II.2). The most common error in (i) was the appearance of a modulus:

- "*$f : D \to \mathbb{C}$ is complex differentiable at $z \in D$ if there exists a limit* $\lim_{h\to 0} \left|\frac{f(z+h)-f(z)}{h}\right|$."
  In this case, $f(z) = \overline{z}$ would be also be differentiable (and the derivative would be a function $f : D \to \mathbb{R}$, or?).

(b) Again, most people got the Cauchy-Riemann conditions right (Theorems II.3.1 & II.3.5). The usual problem: the condition that the partial derivatives are continuous in the sufficient case was missing.

(c) (i) $f(z) = \overline{z}$, $f(z) = \operatorname{Re} z$, $f(z) = \operatorname{Im} z$, $f(z) = |z|$ (see Section II.2) but also $f(x+iy) = 2x+iy$ are all examples of such a function. The students who tried to do it often looked at quite complicated functions.

- "*$f(z) = \frac{1}{\sin z}$ is nowhere differentiable.*"
  This function has a pole at 0 (in fact, at all multiples of π), but on $\mathbb{C} \setminus \pi\mathbb{Z}$ it is holomorphic.
- "*Certainly, we require $u_x \neq v_y$ or $u_y = -v_x$. $f(z) = x^2 + 2xy + y^2 + i(-x^2 + y^2 - 3)$.*"
  This one is differentiable at the origin and thus an example for the next question.

- *"$f(z) = y - i\,x$ is nowhere differentiable since $u_x = 0$, $u_y = 1$, $v_x = -1$, $v_y = 0$. $u_x = v_y$, $u_y = -v_x$ satisfies the R-C equation. So f is continuous complex differentable."*
  This function is $f(z) = i \cdot z$ and thus clearly holomorphic, as the the later of this answer correctly states.
- *"$f(z) = \frac{1}{z^0 - 1} \Rightarrow f'(z)$ does not exist."*

(ii) $f(z) = |z|^2$ (as also $f(z) = |z|^3$, $f(z) = |z|^4$, etc.) is an example as shown in Question 5 on Exercise sheet 2. Some people already struggled with defining a function:

- *"$f(x) = x^2 + i\,y^2 = 0$, the circle that is equal to zero. Differentiating gives $f'(x) = 2x = 0$, $x => 0$ and $f'(y) = 2iy = 0$, $y => 0$. $\therefore$ Is only complex differentiable at $(0, 0)$."*

(d) This is the "unseen" part of Question 1 and requires to use the Cauchy-Riemann equations and then integrate. The final result: the holomorphic functions here are of the form $f(z) = (1 - ia)z^2 + ic$ where $a, c \in \mathbb{R}$ (a is the constant mentioned in the statement of the question, b has to be -1) – which lines up with the non-examinable Section II.5.

- *"Let $u(x, y) = x^2 + 2axy + by^2$, $a, b \in \mathbb{R}^2$. $x = \frac{-2a \pm \sqrt{(2a)^2 - 4\sqrt{b}}}{2}$. Define u which is the real part of a holomorphic function. $(2a)^2 - 4\sqrt{b} \geq 0$, $a^2 \geq \sqrt{b}$, $b \leq a^4$. $\therefore a^4 - b \geq 0$."*

(e) Above, I have splitting this question into the statement (i) and the proof (ii) of the ML-inequality (Lemma III.1.1). Note that f only needs to be continuous and that there are moduli involved here (see the following student's example and first look at the definition of M). In the proof the main question is how to get the modulus "outside" the integral "inside" the integral (because then we are in the real integral case!). 12 students used the proof given in the model solution of the 2007-exam.

- *"The ML-inequality states: $f : \gamma[a, b] \to \mathbb{C}$ is holomorphic and $\gamma : [a, b] \to D$ is a piecewise regular path, then $\int_\gamma f \,\mathrm{d}z \leq ML$ where $M = \max_{z \in I_\Gamma} f(z)$ and $L = \int_a^b |\gamma'(t)| \,\mathrm{d}t$.*
  Proof: $\int_\gamma f \,\mathrm{d}z \leq \left| \int_\gamma f \,\mathrm{d}z \right| = \left| \int_a^b f(\gamma(t))\, \gamma'(t) \,\mathrm{d}t \right| \leq \int_a^b |f(\gamma(t))|\, |\gamma'(t)| \,\mathrm{d}t \leq M \int_a^b |\gamma'(t) \,\mathrm{d}t| \,\mathrm{d}t \leq ML$."

Question 2

Question	maximal marks	mean	standard dev.	median	no. of attempts
(a)(i)	2	1.40	0.60	1	83
(a)(ii)	2	1.86	0.52	2	84
(b)(i)	2	1.22	0.85	1	79
(b)(ii)	5	3.09	1.72	3	74
(c)	2	1.87	0.37	2	82
(d)	3	0.88	0.97	1	65
(e)(i)	1	0.78	0.42	1	67
(e)(ii)	3	1.40	1.27	1	45
overall	20	10.85	4.35	11	85

(a) (i) This is a quite involved definition (see Section I.5), since the path has to be simple closed **and** piecewise regular!

- *"A simple closed contour $\Gamma \subset \mathbb{C}$ is one where the interior $I_\Gamma \subset \mathbb{C}$ for all $\Gamma \subset \mathbb{C}$."*
  I'm guessing that here the student had the Jordan Curve Theorem in mind.
- *"A simple closed contour $\Gamma \subset \mathbb{C}$ is a surjective continuous map $\Gamma : [a, b] \to \mathbb{C}$ where $\Gamma(a) = \Gamma(b) \Rightarrow a = b$."*
  Firstly, a set is not a function. Secondly, if one would take the mapping definition seriously and since the last condition is 'injectivity', one has to remark that a bijective continuous maps $\Gamma : [a, b] \to \mathbb{C}$ unfortunately does not exist (although, surprisingly, if one does not assume injectivity, it is possible, see Wikipedia: Space-filling curve).

(ii) This is the first definition in Section III.1 and was usually done fine.

- *"$\int_\gamma f \, dz = \int_\gamma U(t) + iV(t) \, dt$ for $f : D \to \mathbb{C}$ and $\gamma : [a, b] \to D$."*
- *"The path integral is defined $\int_\gamma f(z) \, d(z) = \int_a^b f(\gamma(t)) \cdot \gamma'(t)$ where $t \in [a, b]$."*

(b) This question was not as clear as I hoped it to be. I had the Criterion for the FTC (Lemma III.2.1) in mind (as 32 of the students did as well), however one can also interpret it to refer to the FTC for path integrals (Lemma III.1.2) (as 42 of you did, some actually stated both or some a mixture). A good thing here is that both lemmata and their proofs are overall approximately equally hard to state and prove, so without further ado, both solutions are okay here (the statement of the respective lemma is part (i) above, its proof part (ii)) and I was very lenient[5] when marking this question. In both lemmata, f is continuous (and necessarily holomorphic) and it is important what kind of paths/contours are involved. In the proof(s), some

[5] Definitely, I did not ask for Lemma III.2.6 and its proof here. Some people really think that a lecturer is a very mean person, don't you?

steps might have been missing (e.g., in the criterion-case, what is the definition of F and/or why is this well-defined).

- *"[...]. If $F(\gamma(a)) = F(\gamma(b))$, $\Gamma \subset D$ is a simple closed contour."*

(c) Cauchy's Theorem (Theorem III.2.2) was mostly done fine. The most common error was that the assumption about the interior was not stated.

- *"Cauchy's Theorem: For a closed contour $\Gamma \subset D$ and $f : D \to \mathbb{C}$ piecewise continuous function, $\int_\Gamma f(z)\,\mathrm{d}z = 0$."*

(d) This was a test for both, if you understand Cauchy's Theorem and the definition of the path integral: The (ridiculous!) $2z^2\cos(z)\sinh(z)$ part of the integrand is holomorphic as product of holomorphic functions thus Cauchy's Theorem applies here. The function $z \mapsto \overline{z}$ is not holomorphic (see Question 1(c)(i)!) thus we have to go back to the definition of the path integral for this part, compare Question 1 on Exercise sheet 4. There is a shortcut using the result of Question 5 on Exercise sheet 4 to get the value of this integral.

- *"Let $f(z) = 2z^2\cos(z)\sinh(z) - 3\overline{z}$. Hence $f'(z) = -(2z^2\sin(z) + 4z\cos(z)) \cdot \sinh(z) + 2z^2\cos(z)\cosh(z) - 3\overline{z}'$. Hence the limit of $f'(z)$ as $h \to 0$ exists, therefore f is holomorphic."*
  This was the typical error, claiming that everything is holomorphic!
- *"$F'(z) = 2z^2\cos(z)\sinh(z) - 3\overline{z}$. $F(z) = \frac{2}{3}z^3\cos(z) - \sinh(z) - 2z^2\sin(z)\sinh(z) + 2z^2\cos(z)\cosh(z)$."*
- "$\cos(z)\cdot\sinh(z) = \left(\frac{\mathrm{e}^{iz}+\mathrm{e}^{-iz}}{2}\right)\left(\frac{\mathrm{e}^{z}-\mathrm{e}^{-z}}{2}\right) = \frac{\mathrm{e}^{iz^2}-\mathrm{e}^{-iz^2}+\mathrm{e}^{-iz^2}-\mathrm{e}^{iz^2}}{4} = 0$. [...]"
- *"Since $2z^2\cos(z)\sinh(z) - 3\overline{z}$ cannot be expanded or factorised or manipulated to find a $(z-5)$ anywhere, then the integral must equal zero."*
  Okay, $z \to \overline{z}$ is not holomorphic (and also not even meromorphic), thus we cannot find a Laurent/Taylor series for the expression. But $2z^2\cos(z)\sinh(z)$ is holomorphic and one finds, in powers of $(z-5)$, that $2z^2\cos(z)\sinh(z) = 50\cos(5)\sinh(5) + (50\cos(5)\cosh(5) + 20\cos(5)\sinh(5) - 50\sin(5) - \sinh(5)) \cdot (z-5) + (2\cos(5)\cdot(10\cosh(5) + \sinh(5)) - 10\sin(5)\cdot(5\cosh(5) + 2\sinh(5))) \cdot (z-5)^2 + \ldots$. Still, the integral of this part vanishes.

(e) This is a nice little exercise where – instead of some cumbersome integration by parts when using the usual real analysis methods (as actually 4 students did in the exam) – we get the stated integrals by simply solving some linear equation system.

  (i) This part often no problem, however the calculating the anti-derivative of the exponential function was sometimes a problem.

  - *"Let $f(z) = \mathrm{e}^z$, then $F'(z) = f(z) = \mathrm{e}^z$, so we have $F(z) = \frac{1}{z}\mathrm{e}^z$."*

  (ii) If the parametrisation $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto t\cdot(a + i\,b)$ of the straight contour was found (and the definition of the path integral in (a)(ii) recalled – there is this factor γ'), everything was fine. Otherwise, there were quite some "remarkable" integration methods:

- *"$\int_0^{a+ib} e^z\,dz = \int_0^{a+i\,b} z = e^{a+ib} - 2$ so $\int_0^1 e^{ax}\cos(bx)\,dx$ and $\int_0^1 e^{ax}\sin(bx)\,dx$ $= \cos^2(bx) + \sin^2(bx) = 1.$"*
- *"$\frac{d}{dx} e^{ax}\cos(bx)\big|_0^1 = a\,(e^{ax}\cos bx) + b\,(e^{ax}\sin bx)\big|_0^1 = a\,(e^a\cos b - 1) + b\,e^a\sin b$ $\Rightarrow \int_0^1 e^{ax}\cos(bx) = \frac{\frac{d}{dx} e^{ax}\cos(bx)\big|_0^1}{a^2+b^2}$. Similarly for $\int_0^1 e^{ax}\sin(bx)\,dx$"*
- *"$\int_0^1 \cos(bx) = \sin b$ so $\int_0^1 e^{ax}\sin(bx)\,dx = e^a\sin b.[\dots]$"*

Question 3

Question	maximal marks	mean	standard dev.	median	no. of attempts
(a)	2	1.60	0.76	2	47
(b)	2	1.48	0.83	2	46
(c)	3	1.11	1.12	1	35
(d)(i)	2	1.35	0.83	2	26
(d)(ii)	2	1.18	0.80	1	28
(d)(iii)	3	1.80	1.38	3	15
(e)	3	1.13	1.24	1	39
(f)	3	1.47	1.23	1.5	30
overall	20	7.45	5.84	6	49

(a) Poor Gauss! Although stated several times (Manifesto & Theorem III.4.2 in the lecture notes, Exercise sheets 6 and 7, and even in the revision lecture), too often done very poorly and I still wonder what a *"non-empty polynomial"* is!

- *"Let $f : D \to \mathbb{C}$ be holomorphic on the domain D. Then it can be deduced that f is uniformly continuous."*
- *"Let $f : D \to \mathbb{C}$ be holomorphic. Then f has zeros in $\mathbb{C}$."*
- *"Gauss' Fundamental Theorem of Algebra says that any polynomial $p(x)$ with zeros is non-constant."*
- *"Gauss' Fundamental Theorem of Algebra: For a polynomial to be non-zero, there exists a constant polynomial."*
- *"Gauss' Fundamental Theorem of Algebra: Every constant polynomial has a nonzero term."*
- *"If $f : D \to \mathbb{C}$ is a non-constant polynomial on D, it has finitely many roots in $\mathbb{C}$."*

(b) I asked for Theorem IV.3.3 here, but stating the local version Lemma III.4.4 was also okay (although, taken seriously, it makes (c) technically more demanding).

- *"Maximum Modulus Theorem: Let $f : D \to \mathbb{C}$ be holomorphic on a domain D, $\forall z_0 \in D$, s.t. $\Gamma = \{z \in \mathbb{C} \mid |z - z_0| < R\} \subset \partial B_R(z_0) \subset D$ and $I_\Gamma = \{z \in \mathbb{C} \mid |z - z_0| < R < B_R(z_0)\} \subset D$. Then if $\int_\Gamma |f(z)\,\mathrm{d}z| = \int_{I_\Gamma} |f(z)\,\mathrm{d}z|$, then f is constant."*

(c) This corollary is implicit in the solution to Question 5 (as well as Question 7) on Exercise sheet 8. So if you have understood that question, this here was no problem.

- *"$\overline{B}_R(0)$ is continuous and bounded, so $f(\overline{B}_R(0))$ is also bounded. Hence $\overline{B}_R(0)$ has a maximum.[...]"*
  Functions are continuous, sets not.
- *"If $f : D \to \mathbb{C}$, $B_R(x_0) \subset D$ and $f(x_0) \leq f(x)$ $\forall x \in B_R(x_0)$ then f is continuous in $B_R(x_0)$."*
  As an exercise: Find a counterexample to this statement!
- *"Proof by contradiction: [...]. But if f is constant, it is not continuous so there is a contradiction."*
  If there is one class of functions that is continuous (although usually boring), then it is the constant functions.

(d) (i) Even if you don't find such a counterexample immediately, looking at (d)(iii) could have given you a hint: Something with a zero in the unit disk. Thus $f(z) = z$, $f(z) = z^2$ etc. are examples if such a counterexample.

  - *"$f(z) = \frac{1}{|z|^2}$ is a counter example, $z = x + i\,y$, $\min |f(z)| = \min \frac{1}{x^2+y^2}$, which happens when $x^2 + y^2 = 1$, i.e., on the boundary of $\partial B_1(0)$. "*
    Is this f holomorphic in the unit disk (even without the modulus)?

(ii) This trick in this question was to find the modulus of e^z which is $\mathrm{e}^{\mathrm{Re}\,z}$. Then, it is easy to conclude that the maximum here is at $z = 1$ while the minimum is at $z = -1$ (well, have a look at picture on page 28 in the lecture notes).

  - *"$f(z) = \mathrm{e}^z = \frac{1}{2\pi i}\int_0^1 \frac{\mathrm{e}^w}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\frac{2\pi}{(1-z)z} = \frac{1}{(1-z)z}$. $0 < |f(z)| \leq 1$, $\max |f(z)| = 1$, $\min |f(z)| = 0$."*

(iii) The most unpopular question on the exam. However, one just has to apply (c) to the function $g(z) = \frac{1}{f(z)}$ (why is this possible?) and immediately gets the statement.

  - *"Suppose it does not take its minimum on the boundary $\partial B_R(0)$, then if it takes it's minimum in $B_R(0)$ then $|f'(z)| = 0$. Picking z arbitrarily $\Rightarrow$ $f' \equiv 0 \Rightarrow f$ constant function. But f can't be constant since at the centre of the ball $f(z) = 0$, but $f(z) \neq 0$ $\forall z \in \overline{B}_R(0)$. Hence $|f(z)|$ takes on its minimum on the boundary $\partial B_R(0)$."*

(e) This statement has been proved in the proof of Gauss' Fundamental Theorem of Algebra (Exercise sheets 6 and 7) and was again given as Question 1 on Exercise sheet 9.

- *"$\left|\sum_{k=0}^{n} a_k\, z^k\right| \leq |a_0| + |a_1\, z| + |a_2\, z^2| + \ldots + |a_n\, z^n| \leq |a_0| + |a_1|\,|z| + |a_2|\,|z^2| + \ldots + |a_n|\,|z^n| \to \infty$ if $|z| \to \infty$."*
  This was the most common error here: If the thing on the right of "$\leq$" goes the ∞, the left side does **not** have to!
- *"$|p(z)| = \left|\sum_{k=0}^{\infty} a_k\, z^k\right| = |a_0 + a_1\, z + a_2\, z^2 + \ldots + a_\infty\, z^\infty| \leq |a_0| + |a_1|\,|z| + |a_2|\,|z^2| + \ldots + |a_\infty|\,|z^\infty| \to 0$ as $|z| \to \infty$."*
- *"$\left|\frac{p(z)}{a_n\, z^n}\right| = 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \ldots + \frac{a_0}{a_n}\frac{1}{z^n} \to 1$ as $|z| \to \infty$. So $\left|\frac{a_n\, z^n}{p(z)}\right| = 1 + \frac{a_n}{a_{n-1}}\, z + \ldots + \frac{a_n}{a_0}\, z^n \to \infty$ as $|z| \to \infty$."*

(f) Using (d)(iii) and (e) one gets a nice proof by contradiction for (a) (so start with: Given a polynomial without zero in $\overline{B}_R(0)$ we have by (d)(iii) that ..., but by (e) we get ... if R is sufficiently large), without using Liouville's Theorem.

- *"$\because p(z) = c\,(z - z_0)\cdots(z - z_n)$, $\because f(z) \neq 0$ $z \in \overline{B}_R(0)$, we can look at $z - z_0$ as $f(z)$. $\because (z - z_0) \in \overline{B}_R(0)$. $\therefore$ by the minimum modulus principle: $|f(z)|$ takes on its minimum on the boundary $\partial B_R(0)$. $|z - z_0| \neq 0$. $|z - z_0| > 0$. $\therefore$ the polynomial has nonzero term."*
  This is the proof for one of the "Theorems" in (a). I am always suspicious of people who can't even be bothered to write "because" and "therefore".

Question 4

Question	maximal marks	mean	standard dev.	median	no. of attempts
(a)(i)	2	1.95	0.22	2	61
(a)(ii)	2	1.91	0.39	2	57
(a)(iii)	2	0.91	0.86	1	46
(b)	2	1.55	0.66	2	55
(c)	3	1.98	0.80	2	54
(d)	3	1.61	1.21	1	46
(e)	2	0.66	0.83	0	35
(f)	4	0.81	1.18	0	21
overall	20	9.29	4.12	8.5	62

(a) (i) The first definition in Section IV.6, mostly done fine.

(ii) The first definition in Section V.2, mostly done fine.

(iii) This can also be found at the beginning of Section V.2. Knowing what a "simple pole" is often was the problem (otherwise, the following would have probably been correct).

- *"*$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}(z-z_0)\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k =$ $\lim_{z\to z_0}\left(\ldots + \frac{a_{-2}}{z-z_0} + \frac{a_{-1}}{1} + a_0\,(z-z_0) + a_1\,(z-z_0)^2 + \ldots\right) = a_{-1}$*. Therefore,* $\operatorname{Res}(f, z_0) = a_{-1} = \lim_{z\to z_0}(z-z_0)\,f(z)$*."*

(b) The statement of Theorem V.2.1 is not about a holomorphic function (but a *"moremorphic"* one;-)); also, formulating the claim was a problem (there is an integral sign, a Γ and an I_Γ!).

- *"Residue Theorem: When a certain set of conditions hold,* $f = \sum_{z\in D}\operatorname{Res}(f, z)$*."*
- *"Assumption:* $f : D \to \mathbb{C}$ *holomorphic.* $\Gamma \subset D$ *is a simple closed contour and* $I_\Gamma \subset D$*. No poles of* f *in* I_Γ*.*
  Claim: $\forall z \in I_\Gamma$, $\int_\Gamma f\,\mathrm{d}z = 2\pi i \sum_{z\in\Gamma}\operatorname{Res}(f, z)$*."*

(c) This is an easy induction (thus, a test if you can do a mathematical argument here unlike in the second example below – and don't write *"let* $n = n+1$*"*), followed by the root test.

- *"*$F(z) = \sum_{n=0}^{\infty} f_n z^n \le \sum_{n=0}^{\infty}(2z)^n = \frac{1}{1-2z}$*."*
  This inequality was a common mistake. Recall that z is a complex variable, thus $F(z)$ as $\frac{1}{1-2z}$ are complex-valued. However, the complex numbers are not ordered, thus this inequality does not make sense (if you don't believe me, evaluate $F(z)$ and $\frac{1}{1-2z}$ for $z = i/10$ or something like that). And by the way, even for real numbers it is not true: $F(-1/10) = \frac{100}{109} \approx 0.91743$ while $1/(1-2\cdot(-\frac{1}{10})) = \frac{5}{6} \approx 0.83333$.
- *"When* $n = 2$*, we have* $f_2 = f_{2-1} + f_{2-2} = f_1 + f_0 = 1 + 1 = 2$*.* $f_2 = 2 \le 2^2$ *which is trivially true for* $n = 2$*. Now suppose it is also true for* $n+1$*. Then* $f_{n+1} = f_{n+1-1} + f_{n+1-2} = f_n + f_{n-1}$ *and* $f_{n+1} \le 2^{n+1}$*. So by induction we have* $f_n \le 2^n\ \forall n \in \mathbb{N}$*. Hence the radius of convergence of* F *is greater than* $\frac{1}{2}$*."*
- *"*$f_{n+1} = f\cdot f_n = (f_{n-1}+f_{n-2})\,f = f\cdot f_{n-1} + f\cdot f_{n-2} = f_n + f_{n-1} \le 2^n + f_{n-1} = 2^{n+1}$*."*

(d) This is about being able to manipulate power series; if you are, then the recursion formula does the rest when adding $z\cdot F(z) + z^2\cdot F(z)$.

- *"*$z\cdot F(z) = \frac{z}{1-(z-z^2)} = z\cdot\sum_{k=0}^{\infty}(z-z^2)^k$*. [...]"*
  What is the radius of convergence here? (And what about these minus-signs, i.e., what was $F(z)$ again?)
- *"Consider the radius of convergence denote as* $\frac{f_i\cdot z^i}{f_{i+1}\cdot z^{i+1}} \le \frac{2^i}{2^{i+1}} = \frac{1}{2}\ \forall i = 0,\ldots,n$*. [...]"*

(e) This checks if you understand the definition of a residue using the power series $F(z)$ – and whether or not you change the summation index from "n" to "k", otherwise you do the following calculation:

- "$\frac{1}{z^{n+1}(1-z-z^2)} = \frac{F(z)}{z^{n+1}} = \frac{\sum_{n=0}^{\infty} f_n z^n}{z^{n+1}} = \sum_{n=0}^{\infty} f_n z^{-1} = \frac{1}{z}\sum_{n=0}^{\infty} f_n$ *and hence* $\text{Res}(\frac{1}{z^{n+1}(1-z-z^2)}, 0) = f_n$."

(f) The ML-inequality establishes that the integral(s) considered here vanish as $R \to \infty$, thus the sum of all the residues equals 0 (compare this with the example(s) in Section V.3). We already know the residue at 0 from (e), but $\frac{1}{1-z-z^2}$ as well as $\frac{1}{z^{n+1}\cdot(1-z-z^2)}$ also has simple poles at the roots of $1-z-z^2$, i.e., at $\frac{-1\pm\sqrt{5}}{2}$. Using (a)(iii) to calculate the residues there, establishes the claim. This was not done very often, usually only the first step was attempted.

- "$f_n = \int_{\partial B_R(0)} \frac{1}{z^{n+1}(1-z-z^2)}\,\mathrm{d}z = \int_{\partial B_R(0)} \frac{1}{z^{n+1}-z^{n+2}-z^{n+3}}\,\mathrm{d}z$
$= -n(n+1)(n+2)\left(z^{-n}-z^{-n-1}-z^{-n-2}\right)\big|_0^{2\pi}$
$\Rightarrow f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$."

Overall

Average: 34.32 ≅ 57.20%   Standard deviation: 10.20 ≅ 17.00%

	minimum	lower quartile	median	upper quartile	maximum
marks out of 60	15	26.5	32	42	59
in percentages	25%	44.2%	53.3%	70%	98.3%

Out of the 87 students, 23 attempted all four questions and 63 attempted three questions (1 student with only 2 questions):

- Average of candidates who attempted 4 questions (23 cand.): 33.9 ≅ 56.5%
- Average of candidates who attempted 3 questions (63 cand.): 34.6 ≅ 57.7%
The popular option was the combination Questions 1 & 2 & 4 (37 cand.), however, the combination Questions 1 & 2 & 3 (24 cand.) was on average slightly more successful by 3.3% (however, for 17 of the 23 students who did all 4 questions, the third question was the one with lowest marks). Two students went for the combination Questions 1 & 3 & 4.

Final grade	[0%,25%[	[25%,40%[	[40%,50%[	[50%,60%[	[60%,70%[
number of students	0	11	22	17	14

Final grade	[70%,85%[	[85%,100%]
number of students	18	5

- In Question 1, I was a bit surprised that part (c) was such a problem. Otherwise, it was okay.
- Based on Question 2, I will in general not trust any (even just real!) integral evaluated by you. This could definitely been better (I guess, right after your schooltime and before coming to university, some of you wouldn't have done the mistakes you made here now).
- Question 3: Some of the answers made me weep! This question tested if you can do some abstract proofs on your own. It was the question with the biggest variation in students' abilities (standard deviation of 5.84!).
- Question 4 parts (c) & (d) on the other hand are more concrete than things in Question 3. It could have been better.
- For each question, there was at least one person who got full marks. Only in Question 3, some students were unable to collect any points.
- The questions that give some indication whether your exam was well, are Question 1(c)(i), Question 2(e)(ii), Question 3(b), (c) & (d)(iii) and Question 4(a)(iii) & (d) – the results in this questions show the highest correlation with the overall exam results.
- And another statistics: Of the 62 students that already did my M41-exam last semester, I am pleased that 27 students did more than 5% better on this exam; sadly, 17 students did more than 5% worse this time.
- I hope you do not misinterpret the "• *"...*"" parts as me making fun of you – they should serve as demonstration what can go wrong in an exam situation, especially if one practises and applies the concepts learned in the unit for the first time in the exam (I do not believe that there are many examples of people who regularly handed in solutions to exercise sheets). Note that some mistakes are interesting and it would have been great to correct them during the exercise session!

Your remarks on the final exam[6]

- *"By the way, most people I spoke to said the exam was hard!"*
- *"As a general comment about the exam, while I think I'd have preferred an exam more similar to previous ones (i.e. more definitions and bookwork-proofs), I appreciate the effort that you went to to come up with original questions, as they'll probably sort the men from the boys!"*

[6] These are all I received so far and I hope that they also reflect your views as well (maybe the second one only if you are one of the 53 male students on this unit).

- *"I thought the exam was challenging, but certainly a good exam. I thought it was written in a way that attacked understanding of the subject material, rather than simply asking the student to recite the material, which was what the majority of past exam papers seemed to do."*
- *"Very nice exam. I actually forgot I was taking an exam at the end, I was drawn in by proving 2(e) and working through 4. Question 3 also looked very interesting!"*

Note that all results and marks stated here are preliminary and therefore might or might not coincide with the eventual final marks! They have neither been checked so far (i.e., whether or not they add up to the sum I came up with and whether or not I have overlooked something) nor has any board/committee looked at them. I thought that giving you meaningful feedback as soon as possible (rather than some weeks from now when we enjoy the hopefully nice summer and you and I can't remember a thing from this exam) is more sensible; so everything here is preliminary and not official, so don't use it against me.