MA30056: Complex Analysis

FAQs (update: May 19, 2009)

Remarks on the lecture notes:

- After marking Question 6 on Exercise sheet 2, the text between Theorem I.5.1. and Lemma I.5.2. has been reformulated to make the distinction between "connected" and "path connected" clearer.
- Following a discussion in one of the drop-in sessions, **Lemma III.1.2** (Fundamental theorem of calculus for path integrals) in the lecture notes has been reformulated. It now reads (and then lines up with its proof):

  Let $f : D \to \mathbb{C}$ be continuous and suppose f has an anti-derivative $F : D \to \mathbb{C}$, i.e., F is holomorphic on D with $F' = f$. Then, the following hold and are equivalent:

  (i) *Let $\gamma : [a, b] \to D$ be a piecewise regular path. Then $\int_\gamma f \, \mathrm{d}z$ only depends on the endpoints $\gamma(a)$ and $\gamma(b)$ of the path γ; more precisely, $\int_\gamma f \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$.*

  (ii) *$\int_\Gamma f \, \mathrm{d}z = 0$ for every closed contour $\Gamma \subset D$.*

- The word "criterium" changed to the word "criterion".

Questions[1] on the lecture notes:

- *Why is the addition of paths defined as it is? I don't understand, it doesn't seem to be a logical definition. For $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$, I would've put* new Ex8Q6

  $$\gamma(t) = \gamma_1(t) + \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t) & \text{if } t \in [c, d], \end{cases}$$

  where $\gamma : [a, b] \cup [c, d] \to \mathbb{C}$. I can't see how to get the definition from here.

  You are actually almost there. So, we want paths to be connected (well, "path-connected" really) and the composition of two paths should again be a path. Thus, we have this condition that we only compose ("add") paths where $\gamma_1(b) = \gamma_2(c)$ (i.e., endpoint of γ_1 is starting point of γ_2). Now, if you use your definition, what happens if, say, $a = 0$, $b = 3$, $c = 2$ and $d = 4$, i.e., $\gamma_1 : [0, 3] \to \mathbb{C}$ and $\gamma_2 : [2, 4] \to \mathbb{C}$? What would be $\gamma(2.5)$?

[1] Ex8Q6 is short for "a question asked in Question 6 on Exercise sheet 7"!

To get rid of this problem, we shift the co-domain of γ_2 such that $[2,4] \to [3,5]$ and therefore we define γ on $[0, 3-2+4] = [0,3] \cup [3,5] = [0,5]$.

And if you have attended the unit M41 (Metric Spaces), then this method is also nice in another aspect: By a result shown in that unit, the image of an interval is connected. So by using intervals as co-domains for paths, one immediately sees that their image is something that is connected (well, one might say that if $\gamma_1 : [0,3] \to \mathbb{C}$ and $\gamma_2 : [7,9] \to \mathbb{C}$ are paths with $\gamma_1(3) = \gamma_2(7)$, then one can define the composition of these paths as you proposed using the co-domain $[0,3] \cup [7,9]$, but then a look at the co-domain would not tell us whether we are actually dealing with a path/something connected or not).

- *Does every simple closed Jordan curve have a regular parametrisation?* **new Ex8Q6**

  No, already the square $\square$ does not have a regular parametrisation (the problem are of course the "corners"). And even if you replace 'regular' by 'piecewise regular', the claim does not hold (e.g., consider some zig-zag-line or jagged line with infinitely many "corners" that is still simple closed). If you think about it then this is another instance of the statement that a real continuous function (here: the path) is not necessarily differentiable (here: smooth) – and the set of points where a real continuous function is not differentiable can be as bad as possible (i.e., there are nowhere differentiable continuous functions!).

- *Going through my notes I am slightly confused by a definition. In particular in Chapter 2, the definitions of continuity and uniform continuity, I don't understand the inequalities,because as z is a complex number what does the expression: $|z-z'| < \delta$, mean, as there is no order to the complex numbers.* **new 20/04**

  Yes, $\mathbb{C}$ has no order, but $\mathbb{R}$ has. And here we look at the modulus of a complex number (the complex number $z - z'$), but the modulus is a map from $\mathbb{C}$ to $\mathbb{R}$. Thus the expression $|z - z'| < \delta$ makes sense since we simply compare real numbers here! (Putting it another way: Since there is no order on $\mathbb{C}$, we first map things into $\mathbb{R}$ where there is an order.)

- *In chapter 3, under 'properties of the path integral', part (iii) parameter invariance, the proof, why is $h(\tilde{a}) = a$, and $h(\tilde{b}) = b$ etc.? Is h supposed to be surjective? If so how does the above follow from it, please explain!* **new 11/05**

  No panic, have a look at (the end of) Section I.5: If we re-parametrize regular paths by a function h, then (using the notation h, a, b, $\tilde{a}$, $\tilde{b}$ etc. used in Section I.5 or Section III.1) $h : [\tilde{a}, \tilde{b}] \to [a, b]$ and h is supposed to be surjective/onto and regular (so, h is continuously differentiable with $h'(x) \neq 0$ for all $x \in [\tilde{a}, \tilde{b}]$). So, by definition, h is a surjective map between two intervals and (by the intermediate value theorem!) is either strictly increasing ($h'(x) > 0$ for all $x \in [\tilde{a}, \tilde{b}]$) or strictly decreasing ($h'(x) < 0$). Now, surjective & continuous & strictly decreasing/increasing (which relies on the ordering of the reals) yields $h(\tilde{a}) = a$ and $h(\tilde{b}) = b$ if $h' > 0$ etc.

- *In the notes, Lemma III.2.1, proof, I cannot understand how you have applied the ML inequality. Seems like there is an h missing? Please could you explain.* **new 17/05**

Okay, so we want to apply the ML-inequality to this integral

$$\left| \int_0^1 (f(z+t\,h) - f(z))\,h\ \mathrm{d}t \right|.$$

Now, first recall the definition of the path integral (see start of Section III.1): You might also write this integral as

$$\left| \int_\gamma (f(w) - f(z))\ \mathrm{d}w \right| \quad \text{where } \gamma : [0,1] \to \mathbb{C}, \gamma(t) = z + t \cdot h.$$

You should now be able to see that the length of this path is $L = |h|$ and the maximum is given by $M = \max\{(f(w) - f(z) \mid w \in \gamma([0,1])\}$. And that's what is written there!

I guesss what confused you was this h in the integral, but that comes from the derivative γ' in the definition of the path integral!

- *I was wondering if for the proof of Cauchy's Theorem (weak version 1), would it be ok to say by the Fundamental Theorem of Calculus we know that the integral is independent of the choice of path, so since the two paths share the same end points, the two integrals are equal? Is that ok, or am I missing something?* **new 05/05**

  So, if I may simplify, the FTC (Theorem III.1.2) tells us that if we have an anti-derivative, then the integral is independent of the choice of path and only the end-points matter (alternatively, that all integrals over closed contours vanish).

  The criterion for the FTC (Theorem III.2.1) tells us that if all integrals over closed contours vanish, then we have an anti-derivative.

  In the proof of Cauchy's Theorem (any version!) we show that this criterion is fulfilled: integrals over closed contours vanish.

  So, unfortunately, you missed the direction of the argument: We show that integrals vanish, thus by the criterion for the FTC we have an anti-derivative (and the FTC holds!).

- *I have a tiny doubt with respect to the proof Cauchy-Goursat. In step 4 you claim:* $\operatorname{diam} T_n \leq \frac{1}{2} L_n$. **new 14/05**
  Where does this come from?

  That is a nice geometrical question! Okay, so L_n is the sum of the side-lengths of the triangle Δ_n. So, let us denote the (complex coordinates of the) vertices of the triangle by z_1, z_2 and z_3. Then $L_n = |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|$ (the side-length is just the Euclidean distance/absolute value between the vertices). Now, how do you determine the diameter of of a triangle? Well, the diameter is given by the maximal side-length $\max\{|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|\}$. W.l.o.g., let us say that $|z_1 - z_2| = \max\{|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|\} = \operatorname{diam} T_n$ (otherwise, renumber the vertices). Thus, we have $L_n = \operatorname{diam} T_n + |z_2 - z_3| + |z_3 - z_1|$. By the triangle inequality, we have $|z_2 - z_3| + |z_3 - z_1| \geq |z_1 - z_2| = \operatorname{diam} T_n$ and we arrive at $L_n = \operatorname{diam} T_n + |z_2 - z_3| + |z_3 - z_1| \geq 2 \operatorname{diam} T_n$.

- *Could you briefly explain (in simple terms!) what the 'cut plane' is, i.e.,* $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. **new 17/05**

  I try to do my best: So, the cut plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is the complex plane with the origin and the negative real line removed. Thus, it is all of the plane with the half-line starting at 0 in direction -1 taken away (on p. 60 in the lecture notes in the lower part of the left picture you may see a sketch of this).
  To point towards its significance: In the pictures of the complex logarithm function on p. 61 in the lecture notes, this half-line we take away is exactly the line where the colour changes abruptly from green-ish to violet-ish. (Recall: we cannot define the logarithm as anti-derivative of $1/x$ in all of $\mathbb{C} \setminus \{0\}$, but we can in the cut plane).

- *The notes under Section III.4.4 Logarithms and multifunctions, first definition, the principal value of the logarithm, is there an i missing before the* $\arg(z)$*?* new 20/04

  You are absolutely right, there is clearly an "i" missing! Thanks for pointing this out! (Corrected in the new version of the lecture notes.)

- *Just wanted to say, i think there is an error in the lecture notes, Theorem IV.2.1 (weierstrass m-test), you have defined k to belong to the naturals, but the sums are starting from* 0. new 20/04

  You are pointing to a blind spot of mine there! Since $\mathbb{N}$ depending on the author/book/article/talk/etc. may or may not include 0, I am actually not very consistent with my notation and $\mathbb{N}$ may or may not include 0. I simply "hope" that it does become clear from the context which – I must admit – might not be very helpful for a student. Sorry for using it in a confusing way, I will try to be more consistent.

- *Why does convergence of power series on an open disk $\Leftrightarrow$ absolute convergence?* new Ex8Q6

  Without going into details, I just give some intuitive reasons why if a power series converges on an open disk, this convergence is actually absolute there.

  Somehow, the geometric series is hiding here, i.e., that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x| < 1$ (and diverges if $|x| > 1$). Now, consider a power series $\sum_{k=0}^{\infty} a_k \cdot z^k$, then we have $\left|\sum_{k=0}^{\infty} a_k \cdot z^k\right| \leq \sum_{k=0}^{\infty} |a_k| \cdot |z|^k$. The root test tells us "something like" (and here I am very unprecise, but I hope this is intuitive; well, I don't tell what I mean by the symbol "$\sim$") $|a_k| \sim 1/R^k$ (where R is this $\limsup$ you get in the root test). So, what we are doing here, is "comparing" the series $\left|\sum_{k=0}^{\infty} a_k \cdot z^k\right|$ with the geometric series $\sum_{k=0}^{\infty} (|z|/R)^k$ and the latter one converges absolutely if $|z| < R$ (here, "comparing" means that the first one converges (absoutely) if the second one does).

  For a more formal proof (actually of the root test), e.g., visit wikipedia:
  http://en.wikipedia.org/wiki/Root_test

- *I don't like Chapter V, I find it hard and confusing!* new 28/04

  Hmm, now you ask for a long answer and a few remarks about Chapter V: The main goal is the Residue Theorem, because the Residue Theorem provides a nice method to evaluate series (like the $\sum_{k\geq 1} 1/k^2$) or integrals (e.g., real integrals of the type $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$).

  Now, what is the Residue Theorem: If you look at the proof, it is a reformulation of the (generalized) homotopy version of Cauchy's Theorem using residues. So what is a residue? If you look at the definition and the remark following it, you should notice that

- they are defined using the Laurent expansion (it is just the coefficient a_{-1}),
- since we have previously shown that the Laurent series expansion is unique, it really makes sense to use this definition,
- if you integrate such a Laurent series term-by-term around a circular contour, then – by the example just before the ML-inequality (Theorem III.1.1) – only the term with coefficient a_{-1} does *not* vanish, thus the alternative integral expression for the residue.

Now, this integral expression enables us to evaluate contour integrals by residues (again, recall that the Residue Theorem is basically only a reformulation of the Homotopy version of Cauchy's Theorem for meromorphic functions).

To make that point clear let us compute the contour integral in Question 2 on Exercise sheet 7 using residues (instead of Cauchy's formulae). So we want to evaluate (for $\alpha > 0$)

$$\int_{|z|=1} f(z)\,\mathrm{d}z \qquad \text{with} \qquad f(z) = \frac{\mathrm{e}^{i\alpha z}}{(z-z_0)^2}.$$

Well, if $|z_0| > 1$, then $f(z)$ is holomorphic in $B_1(0)$, it has no singularity, therefore $\operatorname{Res}(f,z) = 0$ for all $z \in B_1(0)$. So the Residue Theorem yields that this integral vanishes ("we only sum zeros")!

No, if $|z_0| < 1$ (i.e., it is contained in $B_1(0)$), then $f(z)$ has a pole of order 2 at z_0 (and that is the only singularity). So by the Residue Theorem, we already have $\int_{|z|=1} f\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, z_0)$. To calculate this residue, observe (in the step $(\star)$ we use the power series expansion of the exponential)

$$\begin{aligned} f(z) &= \frac{\mathrm{e}^{i\alpha z}}{(z-z_0)^2}\,\mathrm{e}^{-i\alpha\, z_0}\,\mathrm{e}^{i\alpha\, z_0} = \mathrm{e}^{i\alpha\, z_0}\,\frac{\mathrm{e}^{i\alpha\,(z-z_0)}}{(z-z_0)^2} \\ &\overset{(\star)}{=} \mathrm{e}^{i\alpha\, z_0}\,\frac{\sum_{k=0}^{\infty}\frac{(i\alpha)^k\,(z-z_0)^k}{k!}}{(z-z_0)^2} \\ &= \frac{e^{i\alpha\, z_0}}{(z-z_0)^2} + \frac{e^{i\alpha\, z_0}\, i\alpha}{(z-z_0)} + e^{i\alpha\, z_0}\,\frac{(i\alpha)^2}{2!} + e^{i\alpha\, z_0}\,\frac{(i\alpha)^3}{3!}\,(z-z_0) + \dots, \end{aligned}$$

so the residue is $i\alpha\, e^{i\alpha\, z_0}$, and the integral yields (the expected value) $-2\alpha\pi\,\mathrm{e}^{i\alpha z_0}$.

Questions related to and/or on the upcoming final exam:

- *Are 'Optional' questions on the Problem Sheets examinable too?* new 20/04

  Not "directly"; by this I mean: there will certainly be no question on the exam taken from one of the optional question. But clearly, there might be a question where you have to use, e.g., Lemma IV.1.3 (which is proved in Question 7 ExSheet 7) or some argument that is used in one of the optional questions (but certainly also elsewhere). I hope that makes it clear!

- *Is the proof of Cauchy-Goursat examinable? It does seem too long to be on an exam?* **new 11/05**

  I guess all I can say to your question is that this very nice proof is examinable.

- *Is all of III.4.3 non-examinable?* **new 14/05**

  Yes this whole Section III.4.3 about applications in fluid dynamics is non-examinable (I think in some of the previous exams it actually was examinable, though; so, don't be confused by that).

- *When we did theorem 4.4.1, we just saw a brief snippet after easter on an overhead projector and when looking at it today-there doesn't seem to be "proofs" of the equivalent statements, only a direction towards a particular theorem it can be proven by. Does this mean that we would just be required to know this theorem's statements are equivalent???* **new 11/05**

  Yes, we only saw a brief snippet of this Theorem before and after Easter and in the revision session. Why is this so? It is because Theorem IV.4.1 summarises all (well, almost all;-)) theorems in Chapters II-IV. Each implication, e.g., (i)$\Rightarrow$(iv) which happens to be Cauchy's Theorem ("if f is holomorphic then the contour integral vanishes for every simple closed contour in the domain with the interior belonging to the domain"), is one theorem in this unit, but we summarise it here to get an overall picture what holomorphic functions are (and thus what the central objects in complex analysis are). So, we have done all the proofs (okay, one can argue about Cauchy's Theorem). Again: Theorem IV.4.1 summarises "everything" (except numerous applications) we have done up to Easter!

  So, if you have revised up to Theorem IV.4.1, then you either already knew the proof for it (and weren't aware of it), or you have forgotten the statement of some earlier theorems...

- *With regards the impending ma30056 exam, does the examinee require to state "Laurent's Theorem"?, or just to prove it?* **new 19/05**

  You are required to state definitions, theorems etc. as well as prove them.

- *Would the proof of Casorati-Weierstrass be examinable?* new 28/04

  Yes, it is. But the good news is that its proof parallels the proof of Theorem V.1.1(ii) (i.e., the characterization of poles).

- *I have noticed that there are many proof's in our notes, very few of which can be easily disguarded as non-examinable. I am slightly worried that i will need to know them all to secure a good grade (i.e. 65%+) in the exam. Is this the case?* new 05/05

That's hard to answer. It, of course, depends how well you do in the unseen questions and whether or not you make any careless mistakes in questions about definitions. A good part of the exam, however, should test understanding and not so much memorizing abilities. And last but not least, there are always proofs that are more important than others...

- *Thank you for your reply and the revision session you ran last week was quite helpful. I am currently running through the revision checklists you have kindly produced for us and noticed, with respect to your last comment on my previous email, that there is indeed a lot of definitions, theorems and proofs in which to learn, also that your disclaimer indicates that these questions are only a basis for our revision. As a result i am slightly concerned that this unit will be much like metric spaces last semester – very hard.* **new 14/05**
  I realise that short of giving exam hints, there is very little you can do, thus i am emailing you more in respect of feedback than anything else.
  This unit is a very interesting module and i believe that it has been well received by the students taking the course. Despite this, it has been noted by many of my peers that the problem sheets were very complex – excuse the pun, perhaps they are meant to be complex! Also that the lecture notes were very long – 82 pages (when printed, excluding the pages you have mentioned not to print) in comparison to 33 pages for another module we are studying.
  Maybe this says more about other mathematics modules in the department than it does about complex analysis but i thought it was worth mentioning.
  I apologise if this does seem somewhat of a rant, i am very stressed with revision and do not wish it to seem so.
  Once again thank you for your help and support (of which i appreciate there is a lot you have done for us) throughout the semester.

  Thanks for this feedback, I very much appreciate that! I totally understand that you are stressed during the exam period (I guess that's what you are supposed to be right now as a student) and I not at all consider your comments a "rant" – to the contrary. But I would like to comment on some of the points:

  First of all, even if the revision sheets would contain all questions that are on the exam, I would still include the disclaimer. The basis of the exam is the material we did in the lectures and in the exercise sheets, and the "legal (written) proof" what we exactly did there are (in some sense) the lecture notes and model solutions.

  Last year I got the same comments in the feedback about the length of the lecture notes. Personally, I think that this is to a large part a psychological thing here (which, unfortunately, works against me). If one throws out all pics, all exercises, all non-examinable material, change the spacing and the font size from 12pt to 11pt, one is also down to around 33 or so pages. I decided to rather have the lecture notes the way they are now and produce the revision checklist (I didn't do that last year, so that is my reaction on the length comments on the lecture notes) which should help you to condense the material for yourself. However, learning to condense and decide what is important to learn and what not, is one of the skills one should learn at university.

  I will not comment on other units, but even if my exercises are "more complex", I am wondering why the hand-in rate (and not only in my units! so it seems only weakly if at all correlated to "how complex" exercises are) in third year units is so low. Solutions to exercise sheets 6 – 9 (the last four) were handed-in by less than 10 students each, so only around 10% of all students handed something in (and, before that with the exception of the very first one, $\leq 20\%$). Surprisingly or not, most of these around 8 students also come regularly to the drop-in session and ask questions.

To maybe give you some comfort about the upcoming Complex Analysis exam, let me state some statistical details about last years' exam (if this years' is then more difficult or not..., well, we will only know that for sure after next Thursday): The average last year was 62.2% and more than 40% of all students had 70% or more. I hope this gives some motivation for the remaining 7 days that one can do well in a Complex Analysis exam (in a week from now you are done!).

Again, really thank you for your feedback!! It is nice to get it in a more personal way than just from the SAMIS-webpage, and once again good luck in the exam!

- *In the revision session, did you mention how the chapters of our notes will be split across the 4 exam questions?* **new 14/05**

  No, I have not said anything about how the chapters will be split across the 4 exam questions (in fact, sometimes it is not possible, e.g., in last years' exam Question 3(e) can be solved using either Chapter II or III material, Question 4(d) can be solved using either Chapter III or V material).