MA30056: Complex Analysis

Derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$

<u>We recall the following:</u>

Let U be an open set in $\mathbb{R}^n$ and let $\|\cdot\|$ be the usual Euclidean norm. A map $f : U \to \mathbb{R}^m$ is *differentiable at* $p \in U$, if there is a linear map $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - \mathrm{d}f_p(h)\|}{\|h\|} = 0. \tag{1}$$

Given the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, the limits (if they exist)

$$\frac{\partial f}{\partial x_i}(p) = \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t}$$

are the *partial derivatives of* f *at* p.

Note 1: $\exists\, \mathrm{d}f_p \;\Rightarrow\; \exists\, \frac{\partial f}{\partial x_i}(p)$: if f is differentiable at p, then all partial derivatives exist and $\mathrm{d}f_p$ is given by the Jacobi matrix (whose columns are the partial derivatives).

Note 2: The derivative $\mathrm{d}f_{(x_0,y_0)}$ is the "best" linear approximation of f at (x_0, y_0): we have $f(x,y) \approx f(x_0,y_0) + \mathrm{d}f_{(x_0,y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$. Here, $\mathrm{d}f_{(x_0,y_0)} \in M(2 \times 2, \mathbb{R})$ (the *Jacobi matrix*) wherefore the derivative is locally an affine map.

Note 3: We now consider the effect of a linear transformation $A \in M(2 \times 2, \mathbb{R})$:

- The standard basis vectors e_1 and e_2 are mapped to the first and the second column of A.
- A circle is mapped to an ellipse.

We now look at some examples.

Examples. We consider the following three maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$: $f_1(x,y) = (x^2 - xy, 2xy)$, $f_2(x,y) = (x^2 - y^2, 2xy)$ and $f_3(x,y) = (x^2 - xy, -2xy)$.

We calculate the partial derivatives of f_1: $\partial_x f_1$ is given by

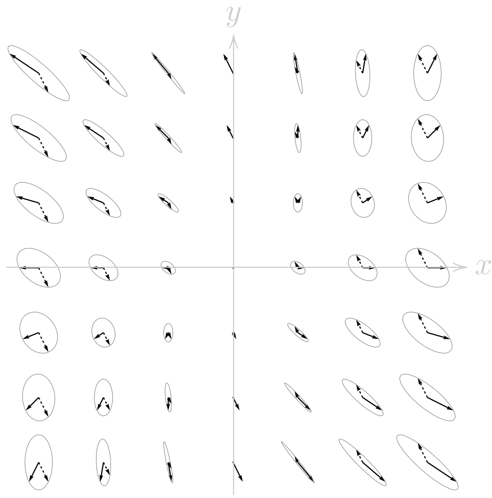
$$\frac{\partial f_1}{\partial x}(x_0, y_0) = \lim_{t \to 0} \frac{\begin{pmatrix} ((x_0+t)^2 - (x_0+t)y_0) - (x_0^2 - x_0 y_0) \\ (2(x_0+t)y_0) - (2x_0y_0) \end{pmatrix}}{t} = \begin{pmatrix} 2x_0 - y_0 \\ 2y_0 \end{pmatrix}.$$

A similar calculation yields $\frac{\partial f_1}{\partial y}(x_0, y_0)$ and we obtain the following Jacobi matrix:

$$(\mathrm{d}f_1)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 - y_0 & -x_0 \\ 2y_0 & 2x_0 \end{pmatrix}$$

Well, actually we would have to check that this is the (unique) linear map we were looking for in Eq. (1).

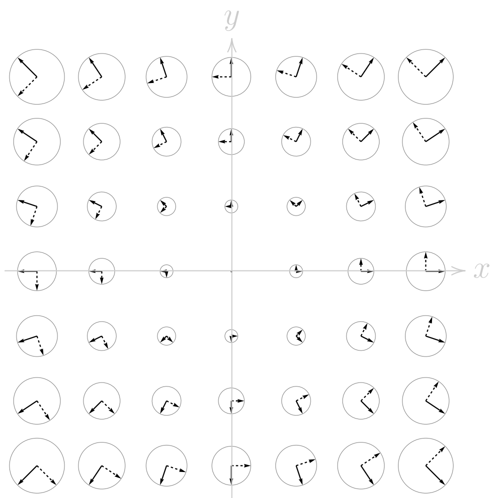
We want to get some geometric intuition what the Jacobi matrix "does". For this, we look at the effect of $(\mathrm{d}f_1)_{(x_0,y_0)}$ on the the standard basis vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the unit circle. The result is the picture on the right: Centred at the point (x_0, y_0), we attach the corresponding ellipse (the image of the unit circle under the Jacobian) and the image of e_1 (black arrow) and e_2 (dashed arrow).



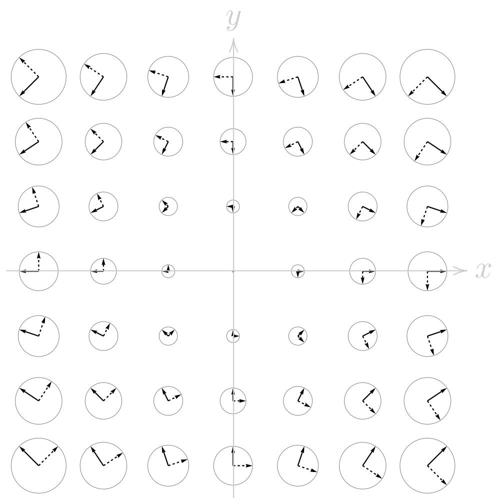
"Geometry" of $\mathrm{d}f_1$.

For f_1 and f_2 we calculate the following Jacobians:

$$(\mathrm{d}f_2)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{pmatrix} \qquad \text{and} \qquad (\mathrm{d}f_3)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 & 2y_0 \\ -2y_0 & 2x_0 \end{pmatrix}$$



"Geometry" of $\mathrm{d}f_2$.



"Geometry" of $\mathrm{d}f_3$.

What do you observe?