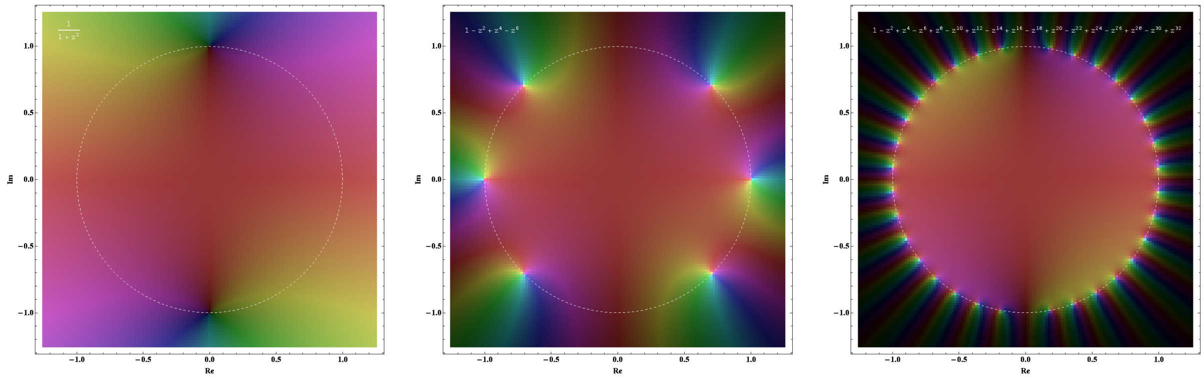
MA30056: Complex Analysis

IV.5. Analytic Continuation & Riemann's Zeta Function
(Not examinable!)

Consider Question 3 on Exercise sheet 8 again[1]: One can check that the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ has radius of convergence $R = 1$. So, the function defined by this Taylor series(!) is certainly not defined for $|z| > 1$.



The function $f(z) = \frac{1}{1+z^2}$ on the left, and two partial sums of its Taylor series around 0, namely up to order 6 in the middle and up to order 32 on the right. Observe that the Taylor series converges in the unit disk to f, but not outside.

We now turn this observation around and ask: Given a holomorphic function on some "small domain", can we find a holomorphic function (and if so, how many[2] such functions) on a "bigger domain" (or at least a domain that has some overlap with the former domain) that extends the function in question.

Definition. If f_1 is holomorphic on a domain D_1 and f_2 is holomorphic on a domain D_2, where $D_1 \cap D_2 \neq \varnothing$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$, then we say that f_2 is a *direct analytic continuation* of f_1 to the domain D_2.

The identity theorem has the following consequences (see [J.W. Dettman, Theorem 4.5.1 & Corollary 4.5.1] for a proof):

Theorem IV.5.1. If f_2 is holomorphic on a domain D_2 and f_3 is holomorphic on D_3 and both functions are direct analytic continuations of a holomorphic function f_1 on a domain D_1 such that $D_2 \cap D_3$ is connected and not empty and $D_1 \cap D_2 \cap D_3 \neq \varnothing$, then $f_2 = f_3$ in $D_2 \cap D_3$.
In particular, if f_1 is holomorphic on D_1 and D_2 is domain such that $D_1 \cap D_2 \neq \varnothing$, then a direct analytic continuation of f_1 into D_2 is unique if it exists. □

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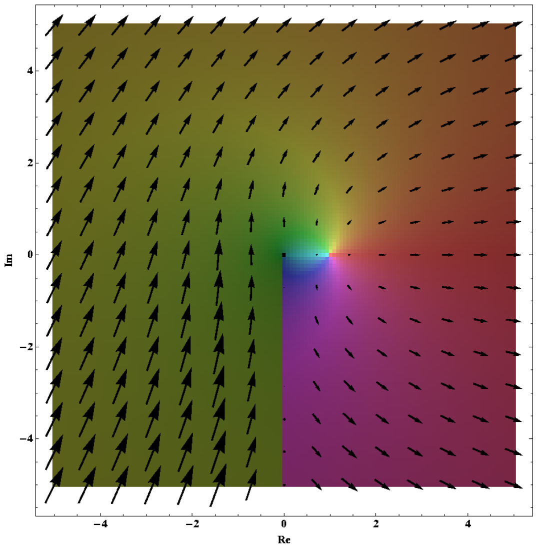
[1] "Find the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ and determine its radius of convergence.
Hint: Do not compute derivatives."

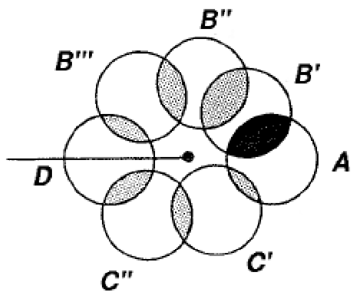
[2] In real analysis, such a question would "not make sense": there are many ways to extend a (real) differentiable function (even if it is C^∞)!

The complex logarithm provides a good example that all assumptions in the previous theorem are necessary: For example, the principal value of the logarithm defined earlier (i.e., on the cut plane $D_2 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$) and the logarithm on the cut plane $D_3 = \mathbb{C} \setminus i \cdot \mathbb{R}_{\leq 0}$ (see right) are direct analytic continuations of the principal value of the logarithm restricted to the domain $D_1 = \{z \mid \operatorname{Re} z > 0\}$, however $D_2 \cap D_3$ is not connected and indeed these two analytic continuations differ in the lower left quadrant.



Vector field and colouring of $\mathbb{C}$ for a logarithm on the cut plane $\mathbb{C} \setminus i \cdot \mathbb{R}_{\leq 0} = \{z \mid \arg(z) \neq -\frac{\pi}{2}\}$.

Furthermore, the complex logarithm also shows that by considering different sequences of (unique!) direct analytic continuations, one might end up with different functions: We begin with the principal value of the logarithm restricted to the disk A, then we can find a unique direct analytic continuation to the disk B'. But this function has a direct analytic continuation to the disk B'', and that function one to the disk B''', which in turn has one to the disk D. But what happens if we consider the sequence of disks A, C', C'' to arrive in D? We end up with a different function, namely[3] we get different branches of the logarithm and the two functions on D differ by a constant of $2\pi i$ (the figure to the right is adapted from [I. Stewart & D. Tall, Fig. 14.9]).



In 8 steps to Riemann's Zeta Function[4]

Riemann's Zeta function is defined by an analytic continuation. One way to establish the Riemann's Zeta function on all of $\mathbb{C} \setminus \{1\}$ is by considering (and proving) the following steps:

(1) We begin with the function defined by an infinite series:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

This series is convergent iff $\operatorname{Re} z > 1$ (and we always use the principal value $n^{-z} = \mathrm{e}^{-z \operatorname{Log}(n)}$ here).

[3] In the Riemann surface picture, we "go up the helix" if we go round the origin counter-clockwise and "down the helix" if we go round the origin in clockwise direction, compare a remark in Section III.4.4 of the lecture notes and a remark to the model soluion of Question 6 on Exercise sheet 7 for "Riemann surface".

[4] This is adapted from Open University Complex Analysis Course Team: "Course M332 Unit 15, Complex Analysis: Number Theory", The Open University Press, Milton Keynes (1975); library: `513.317 OPE`.

(2) We consider the function ζ_1 defined by

$$\zeta_1(z) = z \int_1^\infty \frac{[t]-1}{t^{z+1}}\,\mathrm{d}t + \frac{z}{z-1}$$

(where $[t]$ denotes the integer part of the real number t). We can show that ζ_1 is holomorphic on $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0, z \neq 1\}$ and that $\zeta_1 = \zeta$ for $\operatorname{Re} z > 1$. Thus, ζ_1 is a direct analytic continuation of ζ.

(3) We consider the function ζ_2 defined by

$$\zeta_2(z) = z \int_1^\infty \frac{[t]-t+\frac{1}{2}}{t^{z+1}}\,\mathrm{d}t + \frac{1}{z-1} + \frac{1}{2}.$$

One can show that $\zeta_2 = \zeta_1$ on $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0, z \neq 1\}$.

(4) One can show that ζ_2 is holomorphic on $\{z \in \mathbb{C} \mid \operatorname{Re} z > -1, z \neq 1\}$ and thus ζ_2 is a direct analytic continuation of ζ_1 (and ζ).

(5) We consider the function ζ_3 defined by

$$\zeta_3(z) = z \int_0^\infty \frac{[t]-t+\frac{1}{2}}{t^{z+1}}\,\mathrm{d}t.$$

One can show that ζ_3 is holomorphic *and* $\zeta_3 = \zeta_2$ on $\{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < 0\}$. Hence, ζ_3 is an analytic but not a direct analytic contnuation of ζ.

(6) One establishes that

$$\zeta_3(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

if $-1 < \operatorname{Re} z < 0$. Here, Γ denotes the (complex) gamma function (here is another analytic continuation hidden!).

(7) One shows that the function

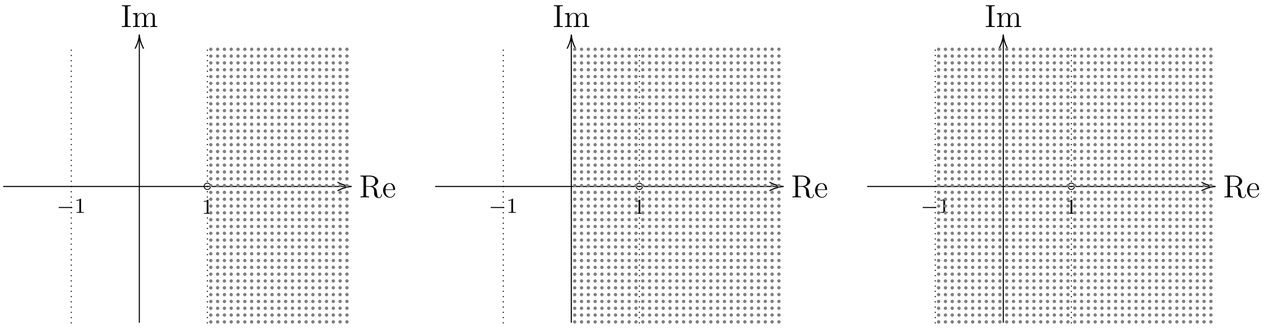
$$\zeta_4(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

is holomorphic on the half-plane $\operatorname{Re} z < 0$ and so ζ_4 is a direct analytic continuation of ζ_3. By step (4), it is also a direct analytic continuation of ζ_2. It follows that the zeta function has been continued analytically onto $\mathbb{C} \setminus \{1\}$.

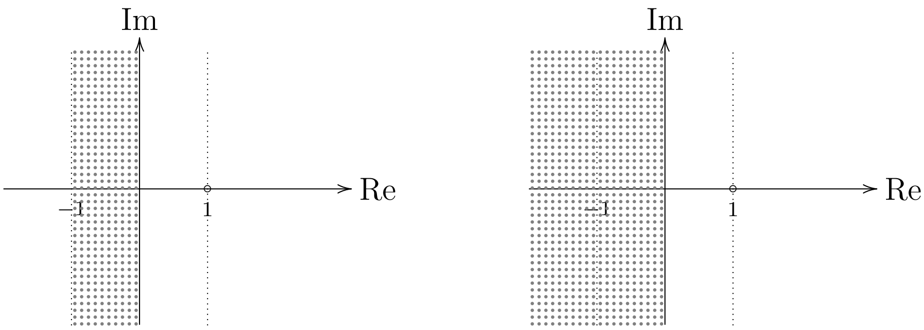
(8) By the "Permanence of Functional Relationships" we have

$$\zeta(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

where ζ now denotes the analytic continuation of ζ in step (1).



The domain on which ζ (see step (1)) is defined on the left, the domain on which ζ_1 (see step (2)) is holomorphic in the middle, and the domain on which ζ_2 (see step (4)) is holomorphic on the right.



The domain on which ζ_3 (see step (5)) is holomorphic on the left and the domain on which ζ_4 (see step (7)) is holomorphic on the right.