MA30056: Complex Analysis

EVALUATING $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$

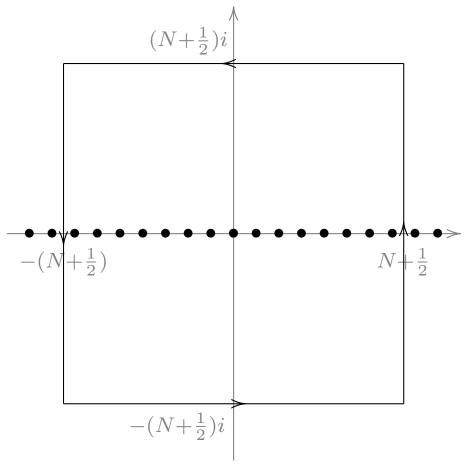
We wish to compute $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Consider $f(z) = \frac{\pi \cot \pi z}{z^2}$. We intend to apply the Residue Theorem to the integral

$$\int_{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} = N + \frac{1}{2}} f \, \mathrm{d}z$$

and then take the limit $N \to \infty$.

The "trick" here is that this integral vanishes for $N \to \infty$ wherefore the sum over all residues equals 0. One then has to relate that sum with the infinite sum we are interested in.



First we determine the residues of f:

(i) The function $g(z) = \pi \cot(\pi z) = \pi \cdot \frac{\cos(\pi z)}{\sin(\pi z)}$ is holomorphic for all $z \notin \mathbb{Z}$ and periodic in $\mathbb{Z}$ (i.e., $g(z+n) = g(z)$ for $n \in \mathbb{Z}$). Its Laurent-series on $B^*_\pi(0)$ is given by[1]

$$g(z) = \frac{1}{z} - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 - \frac{2\pi^6}{945} z^5 - \dots$$

So, g has simple poles of residue 1 at every integer.

(ii) Then $f(z) = g(z)/z^2$ has simple poles for $z = n$, $n \in \mathbb{Z} \setminus \{0\}$, and a triple pole at $z = 0$. Thus

$$\operatorname{Res}(f, n) = \lim_{h \to 0} h f(n+h) = \lim_{h \to 0} \frac{h \cdot g(n+h)}{(n+h)^2} = \lim_{h \to 0} \frac{h \cdot g(h)}{(n+h)^2} = \frac{1}{n^2}$$

for $n \neq 0$ by the periodicity of g. Moreover, the above Laurent-series for g also yields $\operatorname{Res}(f, 0) = -\frac{\pi^2}{3}$.

[1] To obtain the Laurent series, we observe that $\pi z \cdot \cot(\pi z) = \frac{\pi z}{\sin(\pi z)} \cdot \cos(\pi z)$ has a removable singularity at 0. Using the Taylor series for sine and cosine in the relationship $\pi z \cdot \cot(\pi z) \cdot \sin(\pi z) = \pi z \cdot \cos(\pi z)$, one can sucessively obain the coefficients of the Laurent-series for the cotanget.

Now we use the ML-inequality for the above contour integral: Obviously, the length of the contour is $L = 8 \cdot (N + \frac{1}{2})$. For M we observe[2]:

(i) $g(x+iy) = \pi \, \frac{\cos(\pi x)\,\sin(\pi x) \,-\, i\cdot\cosh(\pi y)\,\sinh(\pi y)}{\cosh^2(\pi y)\,\sin^2(\pi x)+\cos^2(\pi x)\,\sinh^2(\pi y)}$

(ii) we have for the "vertical parts" of the contour

$$\left| g\left(iy \pm \left(N + \frac{1}{2} \right) \right) \right| = \left| g\left(\frac{1}{2} + iy \right) \right| = \pi \left| \frac{\sinh(\pi y)}{\cosh(\pi y)} \right| \leq \pi.$$

(iii) we have for the "horizontal parts" of the contour (using $|\cos x|, |\sin x| \leq 1$, $\sinh y < \cosh y$ etc.)

$$\left| g\left(x \pm i\left(N + \frac{1}{2} \right) \right) \right| \leq \pi \, \frac{1+\cosh\left(\pi\left(N+\frac{1}{2}\right)\right)\,\sinh\left(\pi\left(N+\frac{1}{2}\right)\right)}{\sinh^2\left(\pi\left(N+\frac{1}{2}\right)\right)} \overset{N\geq 1}{\leq} 1.28\,\pi$$

Therefore

$$|f(z)| \leq \frac{|g(z)|}{|z|^2} \leq \frac{1.28\,\pi}{(N+\frac{1}{2})^2}$$

for $z \in \mathit{\Gamma}_N = \{x+iy \mid \max\{|x|,|y|\} = N + \frac{1}{2}\}$. The ML-inequality then gives

$$\left| \int_{\mathit{\Gamma}_N} f \, \mathrm{d}z \right| \leq \frac{1.28\,\pi}{(N+\frac{1}{2})^2} \cdot 8 \cdot \left(N + \frac{1}{2} \right) \to 0 \qquad \text{as} \qquad N \to \infty.$$

Thus the Residue Theorem yields

$$-\frac{\pi^2}{3} + 2\sum_{k=1}^{N} \frac{1}{k^2} = \frac{1}{2\pi i} \int_{\max\{|\operatorname{Re} z|,|\operatorname{Im} z|\}=N+\frac{1}{2}} f \, \mathrm{d}z \to 0$$

as $N \to \infty$, that is,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

[2] Using $\sin(x+iy) = \sin x\,\cosh y + i\,\cos x\,\sinh y$ and $\cos(x+iy) = \cos x\,\cosh y - i\,\sin x\,\sinh y$.