MA30056: Complex Analysis

EVALUATING $\int_0^\infty \frac{\mathrm{d}x}{1+x^4}$

We would like to integrate $\int_0^\infty \frac{\mathrm{d}x}{1+x^4}$.

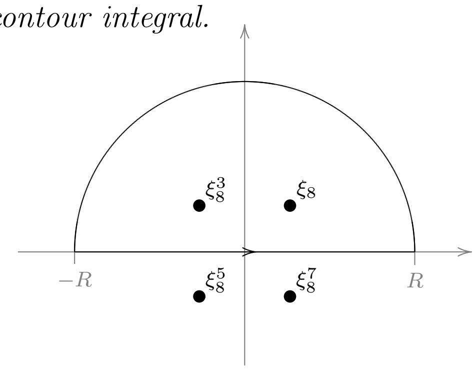
(i) *We associate with the given real integral a related contour integral, of the form* $\int_\Gamma f(z)\,\mathrm{d}z$.

We observe that

$$2\int_0^R \frac{\mathrm{d}x}{1+x^4} = \int_{-R}^R \frac{\mathrm{d}x}{1+x^4}.$$

So, we consider the contour integral $\int_\Gamma \frac{1}{1+z^4}\,\mathrm{d}z$ where $\Gamma = [-R,R] \cup \Gamma(R)$ (with $\Gamma(R) = \{R\,\mathrm{e}^{it}\,|t \in [0,\pi]\}$) is a semicircular contour.

(ii) *We use the Residue Theorem to evaluate the contour integral.*



Note that we have

$$f(z) = \frac{1}{1+z^4} = \frac{1}{(z-\xi_8)(z-\xi_8^3)(z-\xi_8^5)(z-\xi_8^7)}$$

where $\xi_8 = \mathrm{e}^{i\pi/4}$. Thus f has simple poles at ξ_8^k $(k = 1,3,5,7)$, and we calculate the residue

$$\mathrm{Res}(f,\xi_8^k) = \begin{cases} \lim_{z\to\xi_8^k} (z-\xi_8^k)\,f(z), & \text{using Remark at the begin of Section V.2.,} \\ \frac{1}{4\,\xi_8^{3\,k}}, & \text{using the } p/q'\text{-rule (see Question 3 on Exercise sheet 10),} \end{cases}$$
$$= -\frac{1}{4}\,\xi_8^k, \qquad \text{since } \left(\xi_8^k\right)^4 = -1.$$

Then the Residue Theorem yields (for $R > 1$)

$$\int_\Gamma \frac{1}{1+z^4}\,\mathrm{d}z = 2\pi i\,\left(\mathrm{Res}(f,\xi_8) + \mathrm{Res}(f,\xi_8^3)\right)$$
$$= 2\pi i \left(-\frac{1}{4}\right)\cdot\left(\xi_8+\xi_8^3\right) = -\frac{1}{2}\pi i\cdot\sqrt{2}i = \frac{\sqrt{2}}{2}\,\pi.$$

(iii) *We split the contour integral into two parts: a real integral we are interested in, and a complex integral we want to get rid of.*

Obviously,

$$\int_\Gamma \frac{1}{1+z^4}\,\mathrm{d}z = \int_{-R}^R \frac{1}{1+x^4}\,\mathrm{d}x + \int_{\Gamma(R)} \frac{1}{1+z^4}\,\mathrm{d}z = \int_{-R}^R \frac{1}{1+x^4}\,\mathrm{d}x + \int_0^\pi \frac{iR\,\mathrm{e}^{it}}{1+R^4\,\mathrm{e}^{4it}}\,\mathrm{d}t.$$

(iv) *We use the ML-inequality to show that this complex integral becomes arbitrarily small in modulus if we take R to be large enough.*

Using the ML-inequality, we have (by $|z^4+1| \geq |z|^4 - 1$)

$$\left| \int_{\Gamma(R)} \frac{1}{1+z^4} \, \mathrm{d}z \right| \leq \left(\max_{t\in[0,\pi]} \left| \frac{1}{1+R^4\,\mathrm{e}^{4it}} \right| \right) \cdot \pi\, R$$

$$\leq \frac{1}{R^4-1} \cdot \pi R \;=\; \pi\, \frac{R}{R^4-1} \;\to\; 0 \quad \text{as } R \to \infty.$$

Hence the integral $\int_0^\infty \frac{\mathrm{d}x}{1+x^4}$ exists and

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{1}{2}\,\mathrm{PV}\int_{-\infty}^\infty \frac{\mathrm{d}x}{1+x^4} = \pi i\,\big(\mathrm{Res}(f,\xi_8) + \mathrm{Res}(f,\xi_8^3)\big) = \frac{\pi}{2\sqrt{2}}.$$