MA30056: Complex Analysis

II.5 Cauchy-Riemann revisited[1] (Not Examinable!)

We will later see that a holomorphic function $f : D \to \mathbb{C}$ has derivatives of all orders; in particular, it thus has continuous partial derivatives u_x, u_y, v_x, v_y. Using Theorems II.3.1 & II.3.5 (necessary & sufficienten Cauchy-Riemann conditions), we can therefore state: $f : D \to \mathbb{C}$ is holomorphic iff $f = u+iv$ satiesfies the Cauchy-Riemann equations at every point of the domain D. In this case, $f' = u_x + i\,v_x$.

We can state all this using an alternate notation which is more suggestive, but needs some interpretation. We start with defining two operators $\partial/\partial z$ and $\partial/\partial \overline{z}$ by:

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial(iy)}\right)\\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial(-iy)}\right)\end{aligned}$$

There are a few things to be said for this notation:

- The expressions we get "look right": One checks (writing $f = u+iv$) that f satisfies the Cauchy-Riemann equations iff

  $$\frac{\partial f}{\partial \overline{z}} = 0;$$

  in that case, the derivative is given by

  $$f' = \frac{\partial f}{\partial z}.$$

- The "partial derivatives" $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are often easy to calculate: Although the two operators $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are not really partial derivatives with respect to two independent variables (z and $\overline{z}$ are certainly not independent!), when can often pretend that they are and hence just easily "see" what they are. E.g., if f is written as polynomial in z and $\overline{z}$, then we can tell $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ just by looking:

  $$\frac{\partial(z^3\overline{z}^2)}{\partial z} = 3z^2\overline{z}^2, \qquad \frac{\partial(z^3\overline{z}^2)}{\partial \overline{z}} = 2z^3\overline{z}.$$

  (Check that $\frac{\partial z}{\partial z} = 1 = \frac{\partial \overline{z}}{\partial \overline{z}}$) and $\frac{\partial z}{\partial \overline{z}} = 0 = \frac{\partial \overline{z}}{\partial z}$) and that the two operators satisfy the product rule.)

In view of the first point, one would like to say something like "f is holomorphic iff it does not have any $\overline{z}$'s in it"; however, that is too vague to be true (what about $f(z) = |z|^2 = z\overline{z}$?), but the next theorem will give a precise version of this statement.

Please turn over!

[1] This section is adapted from Appendix 7 "The Cauchy-Riemann Equations Revisited" in D.C. Ullrich: Complex Made Simple; AMS, Providence, RI (2008).

We say that a function of two variables $F(z, w)$ is holomorphic if it is holomorphic in each variable separately. If F is a holomorphic function of two variables we will denote the (complex) partial derivatives with respect to the first and second variables by F_1 and F_2 respectively. Then the following theorem is easy to prove if we assume that F is continuously differentiable (in the real sense); and this, actually, follows from the hypotheses on F as given, but that is not easy to show.

Theorem II.5.1. Suppose that F is a holomorphic function of two variables and $f(z) = F(z, \overline{z})$. Then,

$$\frac{\partial f}{\partial z}(z) = F_1(z, \overline{z}), \qquad \frac{\partial f}{\partial \overline{z}}(z) = F_2(z, \overline{z}),$$

and hence:

$$f \text{ is holomorphic iff } F_2 = 0, \text{ in which case } f'(z) = F_1(z, \overline{z}).$$

□

E.g., we can interpret $f(z) = |z|^2$

- either as $f(z) = F(z, \overline{z}) = |z|^2$ (in which case from Question 5 on Exercise sheet 2 we know that $F(z, \overline{z})$ is holomorphic in the first variable only at $z = 0$)
- or as $f(z) = F(z, \overline{z}) = z\,\overline{z}$ (in which case F is holomorphic in both variables, but $F_2(z, \overline{z}) = z$).