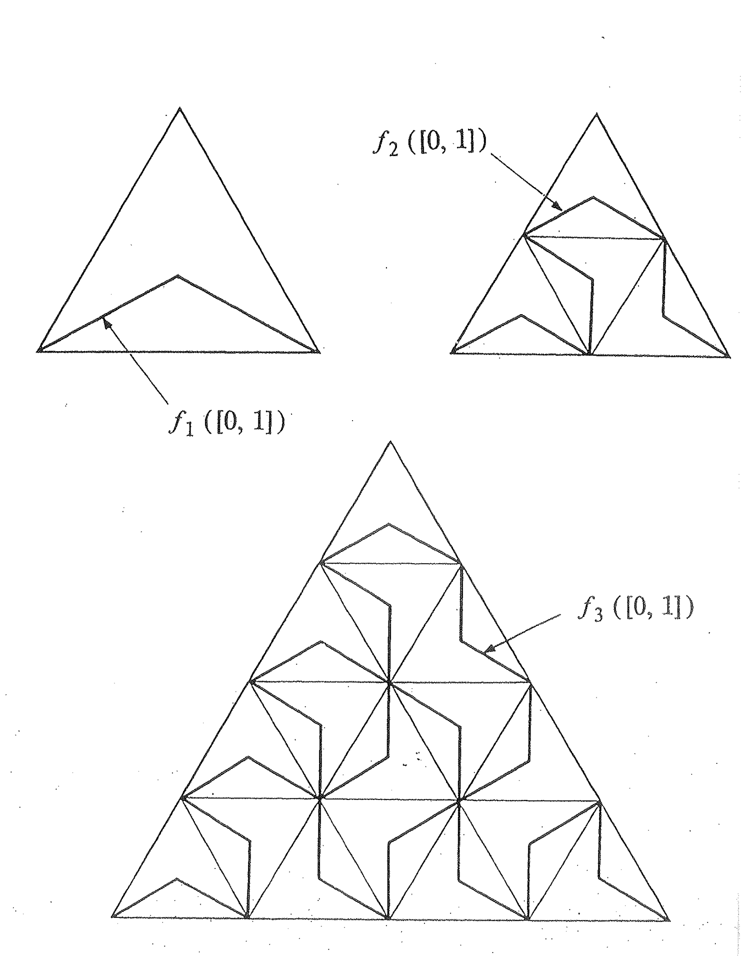
MA30041: Metric Spaces

A SPACE-FILLING CURVE (Not Examinable!)

At the end of the 19th century, Guiseppe Peano made a surprising discovery: There exists a continuous function defined on $[0, 1]$ which maps the interval *onto* a two-dimensional region in the plane, e.g., onto a triangle. Such a function is called a *Peano curve* or *space-filling curve*.

Here is one version: Let Δ be an equlateral triangle in the plane of sidelength 1. We construct a sequence of continuous functions $f_n : [0, 1] \to \Delta$ as follows:



Further elements of the sequence (f_n) are obtained by iterating this procedure. At any particular stage, Δ is divided into a number of congruent triangles, and the part of the curve inside each triangle looks precisely like the image of f_1 and joins two (of the three) vertices of the triangle by a broken line which passes through its centre of mass. To pass to the next stage, we subdivide each triangle into four smaller congruent triangles and insert the more complicated curve which is shown as the image of f_2. As we keep subdividing, the image of f_n fills out more and more of Δ.

We note:

- Let $X = \{f : [0,1] \to \mathbb{R}^2 \mid f \text{ is continuous } \}$. Then, we immediately establish that $(X, d_{\max})$ is a complete metric space where the uniform metric (as usual) is given by
$$d_{\max}(f,g) = \max_{t\in[0,1]} d_E(f(t), g(t)).$$
- Suppose $n \geq m$. Then given $t \in [0,1]$, we can find a triangle which contains both $f_m(t)$ and $f_n(t)$ and whose sidelength is $\left(\frac{1}{2}\right)^{m-1}$. Therefore $d_E(f_m(t), f_n(t)) \leq \left(\frac{1}{2}\right)^{m-1}$, hence $d_{\max}(f_m, f_n) \leq \left(\frac{1}{2}\right)^{m-1}$ and (f_n) is a Cauchy sequence. Let $f \in X$ denote the limit function.
- The previous argument also shows that $f_n([0,1])$ is a $\left(\frac{1}{2}\right)^{m-1}$-net of Δ for all $n \geq m$. Hence $f([0,1])$ is an r-net for any $r > 0$ and thus dense in Δ.
- $f([0,1])$ is the continuous image of a compact set, and hence itself compact. In particular, $f([0,1])$ is closed.
- So, $f([0,1])$ is dense in Δ and closed. Consequently, $f([0,1]) = \Delta$, i.e., f is space-filling!

Good wishes:

for the new year;
for your further studies;
and, of course, for the upcoming exam.

I hope that I was able to explain some of the basic ideas and concepts of Metric Spaces to you and that I was able to share with you some of the excitement about the many results, which generalize the one-dimensional Euclidean world of MA111.

Bernd Sing