MA30041: Metric Spaces

Exercise Sheet 9: Compactness

Please hand solutions in at the lecture on Monday 1st December.

1.) Suppose (A_n) is a sequence of nonempty closed subsets of a metric space (X, d) with $A_{n+1} \subset A_n$ for all n.

(i) Show (by an example) that $\bigcap_{n \in \mathbb{N}} A_n$ may be empty even when (X, d) is complete.

(ii) Show (by an example) that $\bigcap_{n \in \mathbb{N}} A_n$ may be empty even when $\operatorname{diam} A_n \to 0$.

(iii) When one of the A_n is sequentially compact, show that $\bigcap_{n \in \mathbb{N}} A_n$ is nonempty.

(iv) When (X, d) is complete and $\operatorname{diam} A_n \to 0$, show that $\bigcap_{n \in \mathbb{N}} A_n$ is nonempty.

Note: Parts (iii) & (iv) are also called *Cantor's Intersection Theorem*, compare M11.

2.) Let $f : X \to \mathbb{R}$ be a continuous real-valued function on a sequentially compact space (X, d).
Show: f is bounded and attains its bounds, i.e., if $M = \sup f(X)$, $m = \inf f(X)$, then there $\exists x, y \in X$ s.t. $M = f(x)$ and $m = f(y)$.

Note: This is a generalisation of *Weierstrass' Theorem* about the attainment of bounds, compare M11.

3.) (i) Let f be a surjective continuous function from a compact metric space (X, d) into a metric space $(Y, \tilde{d})$. Show that Y is also compact.
Hint: Use Theorem IV.2.

(ii) Let $A_1 = [0, 1] \cup (2, 3]$, $A_2 = [0, 2]$ be subsets of $\mathbb{R}$ with the usual metric. Define $f : A \to B$ by $f(x) = x$ is $x \in [0, 1]$, $f(x) = x - 1$ if $x \in (2, 3]$. Then f is a bijective continuous map from $(A_1, d|_{A_1})$ to $(A_2, d|_{A_2})$. Show, however, that f is not a homeomorphism.

Please turn over!

4.) Suppose that X is nonempty and that (X, d) is a sequentially compact metric space. Let $f : X \to X$ be *contractive*, i.e.,

$$d(f(x), f(y)) < d(x, y) \quad \forall x, y \in X, x \neq y.$$

(i) Show that there exists a unique point $x_0 \in X$ with $f(x_0) = x_0$.
Hint: Show that $\inf \{d(x, f(x)) \mid x \in X\}$ is attained, at x_0 say, and infer that $d(x_0, f(x_0)) = 0$.

(ii) Let $\tilde{x} \in X$. Show that the sequence $\tilde{x}, f(\tilde{x}), f^2(\tilde{x}) = f(f(\tilde{x})), f^3(\tilde{x}), \ldots$ converges to the unique fixed point x_0.

(iii) Let $f : [-\pi, \pi] \to [-\pi, \pi]$ be given by $f(x) = \sin(x) + 1$. What can you deduce about the equation $\sin(x) - x + 1 = 0$? Find (numerically) a root of this equation.

5.) Consider the metric space $(C[0, 1], d_{\max})$. We look at the following set

$$A = \{f \in C[0, 1] \mid f([0, 1]) \subset [0, 1]\}.$$

(i) Show: A is closed and bounded.

(ii) Show: A is not sequentially compact.

6.) (i) Let $\mathbb{R}^2$ be equipped with the *jungle river metric* or *barbed wire metric* d as in Question 2 on Exercise sheet 7.
Show that a closed and bounded subset K of $(\mathbb{R}^2, d)$ is compact iff

- $\forall \varepsilon > 0$ the set $\{x \mid \exists (x, y) \in K \text{ s.t. } |y| \geq \varepsilon\}$ is finite and
- the sets $U_x = K \cap \{(x, y) \mid y \in \mathbb{R}\}$ (for all x) and $V = K \cap \{(x, 0) \mid x \in \mathbb{R}\}$ are compact w.r.t. the Euclidean metric d_E.

(ii) Give an example of a set $A \subset \mathbb{R}^2$ that is closed and bounded (and therefore compact w.r.t. the Euclidean metric), but not compact w.r.t. d.

(iii) Can you find a criterion for a subset K of the X-prickly hedgehog (Y, d) (where $Y = \{0\} \cup (X \times (0, 1])$, see Exercise sheet 1 Question 3) to be compact?