MA30041: Metric Spaces

Exercise Sheet 8: Contractions

Please hand solutions in at the lecture on Monday 24th November.

1.) Let (X, d) and $(Y, \tilde{d})$ be metric spaces and let $f, g : X \to Y$ be continuous maps. Prove: the set $F = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of X. Hence show $f = g$ if F is dense in X.
Hint: Show that F^c is open.

2.) (i) *Heron method for finding square roots*: Let r be a positive real number. Show: $x \mapsto \frac{1}{2}(x + \frac{r}{x})$ is a contraction on $\{x \in \mathbb{R} \mid x^2 \geq r\}$ (well, is it really? If not, why does it not matter here?). Determine the fixed point. What happens if one starts the iteration with some $x_0 \in \{x \in \mathbb{R} \mid x > 0, x^2 < r\}$?

(ii) Show by an example that the completeness assumption in *Banach's Fixed Point Theorem* is important.

3.) (i) *Newton (-Raphson) method for finding roots*: Let f be a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that all its roots are simple (i.e., if $f(x^*) = 0$ for some $x^* \in \mathbb{R}$, then $f'(x^*) \neq 0$). Show: For every root x^* of f,

$$T(x) = x - \frac{f(x)}{f'(x)}$$

is a contraction in some neighbourhood of x^*.

(ii) Calculate the roots of

- $g(x) = x^4 - 3x + 1$ and
- $h(x) = \arctan(x) - x - 1$.

4.) (i) Use Picard's Theorem to find a solution of the intitial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2 + x, \quad x(0) = 1.$$

(ii) Consider the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t + \mathrm{e}^x}, \quad x(\alpha) = \beta,$$

with $\alpha \in [a, b] \subset (0, \infty)$, and show that it has a unique solution $x(t)$ defined on any interval $[a, b]$ and thus on all of $(0, \infty)$.
Hint: Show that for any $t \in [a, b]$ and $x \in \mathbb{R}$ we have $\frac{\mathrm{e}^x}{(t+\mathrm{e}^x)^2} \leq \frac{1}{t} \leq \frac{1}{a}$.

Please turn over!

5.) (i) Let $\mathcal{K}\mathbb{R}^n$ be the space of all nonempty, closed and bounded subsets of $\mathbb{R}^n$. Show that the *Hausdorff metric* d_H, defined by

$$d_H(K_1, K_2) = \max\{\sup_{b \in K_2} \operatorname{dist}(K_1, b), \sup_{a \in K_1} \operatorname{dist}(a, K_2)\}$$

where $K_1, K_2 \in \mathcal{K}\mathbb{R}^n$, is a metric.

(ii) Let $f_0(x) = \frac{1}{3}x$ and $f_1(x) = \frac{1}{3}x + \frac{2}{3}$, and define $f : \mathcal{K}\mathbb{R} \to \mathcal{K}\mathbb{R}$ by $f(K) = f_0(K) \cup f_1(K)$. Show: f is a contraction on $\mathcal{K}\mathbb{R}$.

(iii) Without proof, we are using that $\mathcal{K}\mathbb{R}^n$ is complete w.r.t. d_H. We call the fixed point of the contraction f in (ii) the *(middle third) Cantor set* C. Show: C is bounded, closed, $C' = C$, has empty interior and is uncountable.
Hint: For the empty interior part, use the alternative construction of C by removing subsequently the middle third of each interval (starting from $[0, 1]$). For the uncountability, first show that the set of all infinite 0-1-sequences is uncountable.

6.) Prove the *Implicit Function Theorem on* $\mathbb{R}^2$: Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ be a continous function with $0 < m \leq \frac{\partial f}{\partial x} \leq M$ for some constants m, M, and all $t \in [a, b]$ and $x \in \mathbb{R}$. Then there exists a unique continuous function $x : [a, b] \to \mathbb{R}$ s.t. $f(t, x(t)) = 0$ for all $t \in [a, b]$, i.e., the equation $f(t, x) = 0$ implicitly defines a unique continuous function $x(t)$.
Hint: Show – using the mean value theorem – that

$$T(x)(t) = x(t) - \frac{f(t, x(t))}{M}$$

is a contraction on $C[a, b]$.