MA30041: Metric Spaces

Exercise Sheet 7: Homeomorphsims

Please hand solutions in at the lecture on Monday 17th November.

1.) Suppose that the derivative $f'(x)$ of $f : \mathbb{R} \to \mathbb{R}$ exists at every point $x \in \mathbb{R}$ and $|f'(x)| \leq L$. Show that f is Lipschitz continuous with Lipschitz constant L ($\mathbb{R}$ is equipped with the usual metric).

2.) (i) Consider the X-prickly hedgehog (Y, d) where $Y = \{0\} \cup (X \times (0, 1])$ (see Exercise sheet 1 Question 3). Describe the continuous functions $f : Y \to \mathbb{R}$.

(ii) Recall the definition of the *jungle river metric* or *barbed wire metric* on $\mathbb{R}^2$:

$$d\left((x_1, x_2), (y_1, y_2)\right) = \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1, \\ |x_1 - y_1| + |x_2| + |y_2| & \text{if } x_1 \neq y_1. \end{cases}$$

Describe the continuous functions $f : \mathbb{R}^2 \to Y$ where $(Y, \tilde{d})$ is any metric space and $\mathbb{R}^2$ is equipped with d. How does this compare to the continuous functions $f : \mathbb{R}^2 \to Y$ where $\mathbb{R}^2$ is equipped with the Euclidean metric d_E? Is the identity map $\mathrm{id} : (\mathbb{R}^2, d_E) \to (\mathbb{R}^2, d)$ continuous? What about its inverse?

3.) Let (X, d) be a metric space and define a new bounded metric $\tilde{d}$ on X by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \qquad \forall x, y \in X.$$

Show that id defined by $\mathrm{id}(x) = x$ for all $x \in X$ is a homeomorphism from (X, d) to $(X, \tilde{d})$.

Note: This means that every metric space (even unbounded) is homeomorphic to a bounded metric space.

4.) Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by defined by $d(x, y) = |\mathrm{e}^x - \mathrm{e}^y|$ for all $x, y \in \mathbb{R}$.

(i) Show: d is a metric on $\mathbb{R}$.

(ii) Show: $(\mathbb{R}, d)$ and $(\mathbb{R}, d_E)$ are homeomorphic (where d_E denotes the usual or Euclidean metric on $\mathbb{R}$).

(iii) Are the identity mappings between $(\mathbb{R}, d)$ and $(\mathbb{R}, d_E)$ uniformly continuous?

Please turn over!

5.) Let (X, d) be a metric space, Y be a set and $f, g : Y \to X$ be two maps with the following property: For any uniformly continuous function $h : X \to [0,1]$ (where $[0,1]$ is equipped with the usual metric), we have $h \circ f = h \circ g$.
Show: $f = g$.
Hint: Use Theorem IV.8.

6.) (i) Let (X, d) and $(Y, \tilde{d})$ be two metric spaces and $f : X \to Y$ be uniformly continuous. Show: If $(x_n) \subset X$ is a Cauchy sequence , then $(f(x_n)) \subset Y$ is also a Cauchy sequence.

(ii) Conclude that if f is a homeomorphism s.t. f^{-1} is also uniformly continuous, then (X, d) is complete iff $(Y, \tilde{d})$ is complete.

(iii) Revisit Exercise sheet 3 Question 4: What can you say about the (uniform) continuity of the identity maps between $(\mathbb{N}, d_E)$, $(\mathbb{N}, d_D)$ and $(\mathbb{N}, d_{\text{inv}})$?