MA30041: Metric Spaces

Exercise Sheet 6: Continuous functions

Please hand solutions in at the lecture on Monday 10th November.

1.) Show that the real function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x\,y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x = 0 = y, \end{cases}$$

is not continuous.

2.) (a) Prove Theorem IV.4: *Let (X,d) and $(Y,\tilde{d})$ be metric spaces and $f : X \to Y$. Then, equivalent are:*

(i) *f is continuous on X.*

(ii) $\operatorname{cl} f^{-1}(U) \subset f^{-1}(\operatorname{cl} U)$ *for all subsets $U \subset Y$.*

(iii) $f(\operatorname{cl} A) \subset \operatorname{cl} f(A)$ *for all subsets $A \subset X$.*

(b) Show that corresponding statements for the interior do **not** hold (i.e., "f is continuous iff $\operatorname{int} f(A) \subset f(\operatorname{int} A)$ for all subsets A" is false).

3.) Prove Theorem IV.5: *Let (X,d) and $(Y,\tilde{d})$ be metric spaces and $f : X \to Y$. Then, equivalent are:*

(i) *f is uniformly continuous on X.*

(ii) $\operatorname{dist}(f(U), f(V)) = 0$ *in $(Y,\tilde{d})$ whenever* $\operatorname{dist}(U,V) = 0$ *in (X,d).*

(iii) $\tilde{d}\,(f(x_n), f(y_n)) \to 0$ *as $n \to \infty$ whenever $d(x_n,y_n) \to 0$ as $n \to \infty$.*

Hint: Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). For the proof of (ii) $\Rightarrow$ (iii) show: Let $(a_n), (b_n) \subset Z$ be sequences in a metric space (Z,d') s.t. $d'(a_n,b_n) \geq r > 0$ for all n. Then there is an infinite set $M \subset \mathbb{N}$ s.t. $\operatorname{dist}(\{a_n \mid n \in M\}, \{b_n \mid n \in M\}) \geq \frac{r}{3}$. For this consider the three cases (a) $M = \{m \in \mathbb{N} \mid d'(x,x_m) \leq \frac{r}{3}\}$ is infinite for some $x \in X$, (b) $M = \{m \in \mathbb{N} \mid d'(y,y_m) \leq \frac{r}{3}\}$ is infinite for some $y \in X$, and (c) the sets $\{m \in \mathbb{N} \mid d'(x,x_m) \leq \frac{r}{3}\}$ and $\{m \in \mathbb{N} \mid d'(y,y_m) \leq \frac{r}{3}\}$ are finite for all $x, y \in X$.

Please turn over!

4.) We call a metric space (X,d) is *discrete* if all its subsets are open (and therefore also closed) in X. Obviously, if d is the discrete metric, then (X,d) is a discrete metric space.

(i) Find a function $f : X \to Y$ between metric spaces (X,d), $(Y,\tilde{d})$ that is not continuous but has the property that, for each closed ball $B \subset Y$, the set $f^{-1}(B)$ is closed in X.

(ii) Let (X,d) be a discrete metric space and $(Y,\tilde{d})$ any metric space. Show that any function $f : X \to Y$ is continuous.

(iii) Let (X,d) be a discrete finite metric space and $(Y,\tilde{d})$ any metric space. Show that any function $f : X \to Y$ is uniformly continuous.

(iv) We continue with the metric space $(\mathbb{N}, d_{\text{inv}})$ already studied in Exercise sheet 3 Question 4: Show that $(\mathbb{N}, d_{\text{inv}})$ is a discrete metric space, but find a function on $(\mathbb{N}, d_{\text{inv}})$ that is not uniformly continuous.

5.) (i) For $i \in \{1,\ldots,n\}$ we call the map $p_i : \mathbb{R}^n \to \mathbb{R}$, $(x_1,\ldots,x_n) \mapsto x_i$ the *i-th projection* from $\mathbb{R}^n$.
Show: All projections $p_i : \mathbb{R}^n \to \mathbb{R}$ are uniformly continuous ($\mathbb{R}^n$, $\mathbb{R}$ are equipped with the Euclidean metrics).

(ii) If $f_1 : X \to \mathbb{R}, \ldots, f_n : X \to \mathbb{R}$ are functions, we use the notation $(f_1,\ldots,f_n) : X \to \mathbb{R}^n$ for the map $x \mapsto (f_1(x),\ldots,f_n(x))$.
Show: The map $(f_1,\ldots,f_n)$ is (uniformly) continuous iff all $f_i : X \to \mathbb{R}$ are (uniformly) continuous.

6.) Suppose (X,d) is a metric space and A is a closed subset of X. Show that the function $x \mapsto \operatorname{dist}(x,A)$ defined on X is uniformly continuous.